On the Global Properties of the Model of Developing Countries with Incomplete International Financial Market

Tadashi Inoue
Hiroshima Shudo University
Japan

Abstract

Using a dynasty model of a representative consumer facing upward sloping supply curve of foreign debt, the characteristics of the dynamic paths of capital and consumption are examined both when the international borrowing interest rate increases with foreign debt, and with foreign-debt/capital ratio. The possibilities of bankruptcy and movement toward stationary sate are examined.

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Tadashi Inoue, Faculty of Economic Science, Hiroshima Shudo University
1-1-1 Ozukahigashi, Asaminami-ku, Hiroshima 731-3166, Japan
e-mail; inoue@shudo-u.ac.jp

I. Introduction

This paper tries to show the global dynamic characteristics of the small country open economy facing incomplete international financial market.

The earlier work starts from Stiglitz (1970) employing Uzawa (1968a, 1968b)'s growth model, discusses the two country, two good and two factor economy and the long-run patterns of special??. This is followed by Manning (1981).

What we are concerned with here is a small country who runs trade deficit and borrows from abroad to finance the debt by issuing bond (credit) in the international capital market with borrowing interest become higher, the more borrowing. The higher interest reflects the characteristics of imperfect international credit market.

Hamada (1966) discusses first the case of constant borrowing interest rate and then the case of increasing interest rate with the increase in debt, employing small country optimizing growth model. Bardhan (1967) considers the similar topic of small open country model with borrowing interest depending on the level of debt and with adjustment costs of investment.

Relating to the small country assumption, the constant interest rate requires the time discount rate to be equal to the constant interest rate at the stationary state. To avoid this rather unrealistic assumption, first adjustment costs of investment are introduced, e.g., by Sen and Turnovsky (1989), Turnovsky and Sen (`1991), Turnovsky (1996) and Turnovsky (1996).

The other way to overcome this difficulty is the introduction of recursive preference as tried by e.g., Obstfeld (1982) and Karayalcin (1994)[1/].

The significance foreign debt to the growth of the developing countries have been surveyed by McDonald (1982) as growth-cum-debt models when the importance of foreign debt for finance investment and smoothing consumption overtime are stressed.

Relating to the imperfectness of the international capital markets, as Edwards (1984) is empirical work suggests, the more indebted developing countries tend to pay a premium on their loans from these markets. Reflecting this empirical results, many authors assume that the borrowing interest increases as the increase in debt or foreign capital. Bardhan (1967) is the first who assumes that borrowing interest rate r increases as the increase in foreign capital, k_f. Bhandari et. al (1990) assumes r increases as the increase in foreign debt b, which was followed by Fisher (1995).

Another way to reflect the imperfectness of the international capital market is to assume that borrowing interest r increases as the increase in debt/capital ratio, for the default risk increases as this ratio increases and therefore the lender changes higher interest rate. This approach was adopted by, e.g., Chatterjee et.al. (2003) and Chatterjee and Turnovsky (2005).

We follow the conventional framework of a small country open economy; there exists one good used either for consumption and investment. Production function is neoclassical with ?? primary factors of capital and labor, the home country borrows from abroad to finance current account deficit with borrowing interest rate *r*. Free trade of good is assumed. Factor markets are perfect domestically. We adopt dynasty model. That is, the representative home consumer maximizes the overtime utility function subject to the law of motion of capital and flow budget constraint.

Based on the above framework, we are concerned with *a* global properties of the small country's growth model facing upward sloping borrowing interest rate.

In the next section we deal with the case where the borrowing interest rate increases with foreign debt. Here we show that if the stationary state exists and the initial per capita capital is above certain level, then the economy is globally stable in the sense that per capita capital and per capita consumption converges monotonically to the stationary state.

Otherwise, i.e., if the initial per capita capital is below certain level, the economy goes bankrupt even if the stable state exists, or there exists no optimal path if the exists no stationary state.

In section III, we deal with the case where the borrowing interest *r* goes up with the increase in debt/capital ratio, we obtain the following different conclusions depending on the combinations of parameters when the stationary state exists,

(1) if the initial per capita capital k_0 is above certain level then the economy converges monotonically to the stationary state, but if k_0 is below the level then the economy goes bankrupt (Case-1).

(2) The economy is globally stable irrespective of the level of k_0, but the optimal path of per capita capital and per capita consumption is discontinuous during transitional path (Cases 1-2 and 1-3). Section IV concludes.

II. Borrowing Interest Rate Depends on Debt.

First we consider the case where the borrowing international interest rate *r* increases as the increase in per capita foreign debt *b*, i.e.,

$$r = r(b) \quad \text{with} \quad r'(b) > 0 \quad \text{and} \quad r''(b) > 0\,.$$

Here it is assumed that the rate of increase in interest rate $r'(b)$ increases as foreign debt increases.

The law of motion of domestic per capita capital *k* is expresses as

$$\dot{k} = i \tag{1}$$

when $\dot{k}$ is the time rate of change in k (in general $\dot{x}$ is the time rate of change in variable x.) and i is the per capita investment. To finance trade deficit the home country issues the bond b. It obeys the following law of motion;

$$\dot{b} = r \cdot b - ex \tag{2}$$

where ex is the per capita export, i.e.,

$$ex = f(x) - c - i\,. \tag{3}$$

Here $y = f(x)$ is the labor productivity function, y is the labor productivity, c is the per capita consumption. The function f is neoclassical and satisfies the Inada Condition, i.e., $f' > 0$, $f'' < 0$, $\lim_{k \to 0} f'(k) = t^{\infty}$ and $\lim_{k \to 0} f'(k) = 0$. We treat the case where the home country is a debtor, i.e., $b > 0$.

Let $a = k - b$ be the per capita net wealth which is per capita capital k less per capita debt b. Then from (1), (2) and (3), we obtain the law of motion of a, i.e.,

$$\dot{a} = f(k) - c - r \cdot (k - a)\,. \tag{4}$$

Let $u(c)$ be the felicity function of the representative consumer;

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma} \quad \text{where} \quad 0 < \sigma \neq 1$$[2/].

The representative consumer maximizes

$$\int_0^{\infty} u(c) e^{-\rho t} dt$$

subject to (4) where $\rho > 0$ is the time discount rate, $t = 0$ is the initial time. The consumer takes interest rate r as given although it is dependent on the debt b, since the consumer as microeconomic agent cannot control interest rate, which is determined through macroeconomic conditions. Let $W(k_0) = \max_{\dot{a}} \int_0^{\infty} u(c) e^{-\rho t} dt$ be the value function which is the maximized value of the objective function subject to the law of motion of the net wealth a.

Then the above maximization problem is solved by forming the following current value Hamiltonian;

$$H = u(c) + \lambda(f(k) - c - r \cdot (k - a)) \tag{5}$$

and obtaining the first order conditions

$$c^{-\sigma} = \lambda\,, \tag{6}$$

$$f'(k) = r(b)\,, \tag{7}$$

$$\dot{\lambda} = \rho\lambda - r\lambda\,, \tag{8}$$

and the transversality condition (TVC)

$$\lim_{t \to \infty} \lambda a e^{-\rho t} = 0\,. \tag{9}$$

From (6), (7) and (8), we obtain

$$\dot{c} = \sigma^{-1} c(\rho - f'(k)). \tag{10}$$

From (7) we obtain

$$db/dk = f''/r' < 0. \tag{2}$$

Fig. 1 shows the function $b = b(k)$.

Fig. 1

Here at $k = k_M$, $f'(k_M) = r(0)$ holds. That is, k_M is the maximum value of k where the marginal of product of capital equals the world borrowing interest rate $r(0)$ of no foreign borrowing.

As for net wealth $a = k - b$, we observe

$$da/dk = 1 - db/dk > 1$$

and the amount of debt payment $g = g(k) = r \cdot b$, it follows that

$$g'(k) = r' \cdot b \cdot db/dk + r \cdot db/dk < 0$$

with $g(k_M) = 0$. In short we obtain the following Fig. 2 of $f(k)$ and $g(k)$ curves. As seen in Fig. 2, let $k = \underline{k}$ where $g(k) = f(k)$ holds. Then $f(k) > g(k) \Leftrightarrow \underline{k} < k$.

Fig. 2

Recalling $\dot{a} = f(k) - c - b(k)$, $\dot{a} > 0 \Leftrightarrow \dot{k} > 0$ from $da/dk > 1$, and (10), we obtain the phase diagram of (k, c) for this case. There exist two subcases depending on the existence of the stationary point E. In Fig. 3, $\dot{k} = 0$ curve is equal to $\dot{a} = 0$ curve. That is $\dot{k} = 0 \Leftrightarrow c = f(k) - g(k)$. Then at $k = \underline{k}$, $\dot{k} = 0$ curve intersects with the horizontal axis. From (10) with $\dot{c} = 0$, let $k = \bar{k}$ be the value of k where $f'(k) = \rho$. Then $\dot{k} = 0$ curve is the upward sloping curve and $\dot{c} = 0$ curve is the vertical line with $k = \bar{k}$. Furthermore $\dot{k} > 0 \Leftrightarrow (k, c)$ is in the lower-right of $\dot{k} = 0$ curve, and $\dot{c} > 0 \Leftrightarrow (k, c)$ is on the left-hand-side of $\dot{c} = 0$ vertical line. First we consider

Case 1 (Fig. 3)

where $\underline{k} < \bar{k}$ so that the stationary point E exists.

Fig. 3 (Case 1)

As seen from the phase diagram (Fig. 3), the stationary state E is characterized by $\dot{c} = 0$, i.e., $\rho = f'(k)$ and $\dot{k} = 0$, i.e., $c = \bar{c} = f(\bar{k}) - g(\bar{k}) > 0$ where $\rho = f'(\bar{k})$. As seen, if $\underline{k} < k_0 < \bar{k}$, i.e., if the initial per capita capital k_0 is lager than $\underline{k}$ but less than $\bar{k}$ then the economy converges monotonically to the stationary state $E(\bar{k}, \bar{c})$ with both k

and c are increasing. Similarly if $\underline{k} < k_0\ (\leq k_M)$ then again the economy converges monotonically to the stationary state $E(\bar{k},\ \bar{c})$ with both k and c are decreasing.

However if $k_0 < \underline{k}$, then either k becomes zero and c becomes finite in a finite time $T < \infty$ as drawn by the curve *ab*, or k becomes zero and c becomes infinite as $t \to \infty$ as shown by $a'b'$. For both cases TVC is seen to be violated as discussed in the following. If the optimal path of (k, c) follows *ab*, then it arrives at b in a finite time $t = T < \infty$. Then the transversality condition (TVC) for this kind of free finite terminal time is that the current value Hamiltonian H must vanish at $t = T$ [3/], i.e.,

$$H = 0 \quad \text{at} \quad t = T. \tag{11}$$

However as seen from (11) $k \to 0 \Leftrightarrow f'\uparrow \Leftrightarrow r \to +\infty$ and hence $k \to 0$ implies $g(k) = r \cdot b \to +\infty$. Then in view of (5), $\lambda = 0$ must hold at $t = T$, and hence from (6), $c \to \infty$ as $t \to T$, a contradiction. This implies such path as *ab* violates TVC.

Next we also show such path as $a'b'$ also violates TVC (9). Here we define NPG (No-Ponzi-Game) condition which is obtained from TVC (9),

$$\text{NPG:} \quad \lim_{t \to \infty} b(t) e^{-\int_0^t r(\tau) d\tau} = 0 \tag{12}$$

[4/]

However along $a'b'$, NPG is seen to be violated.[5/] Intuitively NPG condition implies that the country cannot borrow more and more from abroad for financing foreign debt. Violation of NPG shows that along $a'b'$, the home country goes into bankruptcy.

In short even if the stationary state E exists, if the initial per capita capital k is below $\underline{k}$, then the economy cannot move forward the stationary state. In stead the home country goes into bankruptcy.

Next we consider

Case 2 (Fig. 4)

where $\bar{k} \leq \underline{k}$. For this case there exists no stationary state. Henceforth we assume that once $k = k_M$ is reached then the home country lends abroad by receiving interest rate $r(0)$. This implies that k never exceeds k_M.

Fig. 4 (Case 2)

For this case, just Case 1 with $k_0 \leq \underline{k}$, we observe there exists no optimal path. In fact, if $k_0 \leq \underline{k}$ holds and stationary point are as A and A', then the solution paths ABk_M and $A'B'$ are indicated by the phase diagram. However by sticking to points D and D' which are respectively above points A and A' forever, the larger amounts of value function is realized from this suboptimal paths. This implies that such paths as ABk_M and $A'B'$ are not optimal.

Furthermore if starting points are a and a', then b is reached in a finite time $T<\infty$ and $k\to 0$ and $c\to+\infty$ as $t\to+\infty$ hold by following $a'b'$, both violating TVC just as Case 1 with $k_0 \le \underline{k}$. In short for this case, there exists no optimal path.

III. Interest Rate Increases with debt/capital Ratio

Next we consider the case where the borrowing interest rate r increases as the debt/capital ratio increases. That is

$$r = r(b/k) \quad \text{with} \quad r'(b/k) > 0 \quad \text{and} \quad r''(b/k) > 0$$

where b is the per capita bond (debt) issued by the home country and k is the amount of the home per capita capital. This assumption is made by e.g., Chatterjee, Sakonlis and Turnovsky (2003) and Chatterjee and Trunovsky (2005). The law of motion of capital is (1), i.e.,

$$\dot{k} = i. \tag{1}$$

As before we obtain

$$\dot{a} = f(k) - c - r\cdot(k-a) \tag{4}$$

the law of motion of per capita net wealth a.

Here we assume again that the consumer takes the interest rate r as given, although it is dependent on debt/capital ratio. Then again the above maximization problem is solved by forming the current value Hamiltonian (5) and obtaining the first order conditions (6), (7), (8) and TVC (the transversality condition) (9) .

From (6), (7) and (8), we obtain again (10). From (7) with $r = r(b/k) = r((a-k)/k)$, we obtain

$$da/dk = (-f''{\cdot}k + r'{\cdot}a/k)/r'. \tag{13}$$

To obtain the definite results, we specify r and $f(k)$ to be

$$r = r(0) + e^{\beta n} - 1 \tag{14}$$

where $\beta > 0$, $n = b/k$ and $r(0)$ is the international interest rate with no debt.[6/] Here $0 < r(0) < 1$ is assumed. $f(k)$ is of Cobb-Douglas so that

$$f(k) = k^{\alpha}, \quad 0<\alpha<1$$

where α is the capital share. With these specification we obtain from (13)

$$da/dk > 0 \Leftrightarrow (1-\alpha)r + r'{\cdot}(1-n) > 0 \Leftrightarrow n < n^*.$$

where $n = n^*$ is the value of n such that

$$(1-\alpha)r(n) + r'(n)(1-n) = 0. \tag{2}$$

Fig. 5

In fact we observe from Fig. 5 that the slope of $r'(n)\cdot(n-1)$ curve is $r' + (n-1)r'' = r' + (n-1)\beta r' = (1+\beta(n-1))r'$ and that of $(1-\alpha)r(n)$ is $(1-\alpha)r'(n)$, and hence the former's slope is higher than the latter's slope if $n>1$ as drawn in Fig. 5, implying two curves intersect once at $n = n^* > 1$.

Now recalling $f'(k) = r(n)$, (7) to imply $dn/dk < 0$, we obtain Fig. 6 where $k = k^*$ is defined to be the value of *k* such that $f'(k) = r(n^*)$ hold. As seen from Fig. 6, $da/dk > 0 \Leftrightarrow k > k^* \Leftrightarrow n < n^*$.

Fig. 6

Here we note that at $n \leq 1$, $a \geq 0$, and for $n > 1$, $a < 0$ follow. The maximum value of *k*, k_M is defined implicitly by $f'(k) = r(0)$, and hence at $k = k_M$, $a = k_M$ must hold. Furthermore by conclusion, it is shown that $a \to 0$ as $k \to 0$.[7/]

Fig. 7

Next we examine the function,

$$g(k) = r(n(k)) \cdot (k - a(k)), \tag{15}$$

the amount of the debt payment. By rewriting *g*(*k*) as

$$g(k) = r(1 - a/k) \cdot (k - a)$$

we obtain

$$g'(k) = r' \cdot a \cdot b / k^2 + r - a(r + r' \cdot b/k)$$

where a' is obtained from (7), (13) and $f(k) = k^\alpha$ as $a' = (1-\alpha)r/r' + a/k$.

Then $g'(k)$ is rewritten as

$$g'(k) = r \cdot (n\alpha - (1-\alpha)r/r'). \tag{16}$$

Then

$$\operatorname{sgn} g'(k) = \operatorname{sgn}(n\alpha r' - (1-\alpha)r).$$

Fig. 8

Since the slope of $n\alpha r'$ curve is $\alpha(1 + n\beta)r'$ from $r'' = \beta r'$ and the slope of $(1-\alpha)r$ is $(1-\alpha)r'$ in Fig. 8, the former is always higher. Hence there exists $n = \hat{n}$ such that $n\alpha r' = (1-\alpha)r$ holds and

$$n > \hat{n} \Leftrightarrow n\alpha r' > (1-\alpha)r \Leftrightarrow k < \hat{k} \Leftrightarrow g' > 0$$

where $\hat{k}$ is defined implicitly by $f'(k) = r(\hat{n})$. Then we obtain *g*(*k*) curve as shown in Fig. 7. Here at $k = k_M$, $g(k) = 0$ follow from $b = 0$ at $k = k_M$. Furthermore recalling $a \to 0$ as $k \to 0$, we observe $g(k) \to 0$ as $k \to 0$.[8/] Next by calculation we observe

$$f(k) > g(k) \Leftrightarrow n < 1/\alpha \Leftrightarrow k > \underline{k}$$

where $\underline{k}$ is defined implicitly by $f'(\underline{k}) = r(1/\alpha)$.[9/] This is shown in Fig. 7. (Here we note that although $\underline{k} < \hat{k}$ is assumed in Fig. 7, it is not so. In fact the rest of the arguments goes through irrespective of $\underline{k} \overset{<}{\geq} \hat{k}$.)

Now let $k = \bar{k}$ be the value of k such that $f'(k) = \rho$ and hence $r = \rho$ holds.

Now we consider the possibility of the existence of the optimal path (k, c), employing the phase diagram technique. First we consider the case where such an optimal path exists under an appropriate choice of (k, c). This is equal to the case where the $\dot{c} = 0$ curve (the vertical line $k = \bar{k}$ in $k - c$ plane) and the $\dot{a} = 0$ curve (the upward sloping curve starting from $(\bar{k}, 0)$ in $k - c$ plane) intersect and the point of intersection $E(\bar{k}, \bar{c})$ is the stationary point of (k, c) where the optimal path (k, c) converges as $t \to \infty$. This case if

Case 1 $\underline{k} < \bar{k}$ (i.e., $\beta - r_0 + 1 < e^{\beta/\alpha}$)

as shown in Fig. 9.1 through 9.3. Case 1 is further subdivided into Case 1-1 ($k^* < \underline{k}$), Case 1-2 ($\underline{k} < k^* < \bar{k}$) and Case 1-3 ($\bar{k} < k^*$) corresponding to Fig. 9.1, 9.2 and 9.3 respectively. Here recalling $da/dk > 0 \Leftrightarrow k > k^*$, we observe that $\dot{k} > 0 \Leftrightarrow \dot{a} > 0$ for $k > k^*$, and that $\dot{k} < 0 \Leftrightarrow \dot{a} > 0$ for $k < k^*$.

This implies that the horizontal arrowed line showing the ??? of $\dot{k}$ is reversed on the both sides of $k = k^*$ vertical line for all three cases as shown in Fig. 9.1 through 9.3. Now we can draw the solution paths of (k, c) based on the phase diagram for each case. For all three cases, the optimal path converging to $E(\bar{k}, \bar{c})$ is a saddle point path.

For Case 1-1 (Fig. 9.1) the optimal path of (k, c) starting from $k_0 \geq \underline{k}$ converges to E, but the solution path starting from $k_0 < \underline{k}$ is the one through which the home country goes bankruptcy eventually as shown later.

For Case 1-2 and 1-3, the economy can reach to the stationary state E starting from <u>any</u> initial k_0, but as seen from Fig. 9.2 and 9.3 the optimal path (k, c) becomes <u>discontinuous</u> at $k = k^*$. More specifically for Case 1-2, the economy starting from $k_0 < k^*$ can enjoy higher consumption for a while but at $k = k^*$, the consumption must go down and the keeps continuously increasing till $k = \bar{k}$.

Similarly for Case 1-3, the economy starting from $k_0 > k^*$ can enjoy higher consumption for a while till $k = k^*$, but at $k = k^*$, the consumption goes down and then keeps continuously decreasing till $k = \bar{k}$. For the last two cases 1-2 and 1-3, the level of consumption c till k reaches k^* must be chosen so that the objective function $\int_0^\infty u(c) e^{-\rho t} dt$ is maximized subject to (4).[10/]

Here Case 1-1, $k^* < \underline{k}$ is equal to $(\alpha - \beta) e^{\beta/\alpha} > \alpha(1 - r_0)$ while in Cases 1-2 and 1-3

$\bar{k} < k^*$ is equal to $(\alpha - \beta) e^{\beta/\alpha} > \alpha(1 - r_0)$.[11/]

Furthermore in Case 1-2 (resp. Case 1-3) $k^* < \bar{k}$ (resp. $k^* > \bar{k}$) is equal to $a'(k) > 0$ (resp. $a'(k) < 0$) at $k = \bar{k}$ (i.e., $\rho = r$), which is equal to $(1-\alpha)r + r'(1-n) > 0$ (resp. < 0) at $k = \bar{k}$ (i.e., $\rho = r$) or $(1-\alpha)\rho / \beta >$ (resp. $<$) $(\bar{n} - 1)e^{\beta \bar{n}}$ where $n = \bar{n}$ is the value of at $k = \bar{k}$, i.e., $\bar{n} = \dfrac{1}{\beta}\log(\rho + 1 - r_0)$ from (14) with $\rho = r$.

Recalling here that Case 1, $\underline{k} < \bar{k}$ is equal to $\rho - r_0 + 1 < e^{\beta/\alpha}$, we note that the stable saddle point path of (*k*, *c*) is possible only if the discount rate ρ is law, initial world interest rate (interest rate without foreign borrowing) r_0 is high or β / α (=increasing rate of borrowing interest rate/capital-share) is high.

In Case 1-1 where $k^* < \underline{k}$ (i.e., $(\alpha - \beta)e^{\beta/\alpha} > \alpha(1 - r_0)$ implying $\alpha > \beta$) as seen from Fig. 9.1, if initial per capita capital is less than $\underline{k}$, then the solution path of (*k*, *c*) does not converge to the stationary state $(\bar{k}, \bar{c})$ but to (k^*, ∞) , and the home country goes into bankruptcy because she accumulates foreign debt. That is, along this latter path of (*k*, *c*), NPG(No-Ponzi-Game) condition is violated again as seen by path $a'b'$ in Fig. 3 (Cases if II).

In short, the characteristics of the solution paths of (*k*, *c*) when the optimal stable saddle point path exists (i.e., Case 1) are, first for Case 1-1, the possibility of the accumulation of foreign debt and the resulting bankruptcy when the initial k_0 is low, and second for Case 1-2 and 1-3, the discontinuity of the optimal path.

Next we consider

Case 2. $\bar{k} \le \underline{k}$ (i.e., $\rho - r_0 + 1 \ge e^{\beta/\alpha}$), the nonexistence of the stationary state. This case is further subdivided into Case 2-1. $k^* \le \underline{k} \le \bar{k}$, Case 2-2 $k \le k^* \le \bar{k}$ and Case 2-3. $\underline{k} \le \bar{k} \le k^*$, as shown respectively in Fig. 10.1, 10.2 and 10.3.

Here first we note that on the left (resp. right) side of the vertical line of *k**, $\dot{k} > 0$ (resp. < 0) $\Leftrightarrow \dot{a} < 0$, as the Case 1, $\underline{k} < \bar{k}$. Then we can obtain phase diagram for all three cases.

First we consider

Case 2-1 (Fig. 10.1)

Just as in Case 1-1, if the path starts from point *a* or a' , then *c* goes to infinity as $k \to k^*$ and $t \to \infty$. This implies that the home country goes bankruptcy, violating NPG just as Case 1-1. The path starting from point *b* arrives at *B* in a finite time $t < +\infty$. Then the transversality condition (TVC) for this kind of free terminal time is that the current value

Hamiltonian H must vanish at $t = T$ again just as path *ab* in Case 1 of II (Fig. 3), i.e.,

$$H = 0 \quad \text{at} \quad t = T\,. \tag{11}$$

Then recalling $c = 0$ and $\dot{a} = f(k) - c - r(k-a) > 0$ at *B*, we must set $\lambda = 0$ at $t = T$ from (5). However then from (6) $\lambda \to +\infty$ as $t = T$, a contradiction.

Here recalling that once $k = k_M$ is reached and the home country owes no debt, i.e., $b = 0$, then she receives the fixed world interest rate *r*(0) when she turns into creditor, i.e., $b < 0$. Then we obtain as a first order condition

$$f'(k) = r(0) \tag{17}$$

if $b < 0$. Then (17) implies $k = k_M$ for $b < 0$ and hence $\dot{k} = i = 0$. This further leads to

$$\dot{a} = -\dot{b} = f(k_M) - c + r(0)\cdot(-b)$$

from the flow budget constraint (4) where $-b > 0$ is the asset held by the home consumer.

Here we next consider the path starting from b' in Fig. 10.1. As discussed the above, once it reaches *d* on the vertical line after passing k_M, it moves down along $d\,k_M$ toward (k_M, 0), and reaches there in a finite time *T*. However then the TVC, $H > 0$ at $t = T$ (11) is violated again. Here we note that such paths as starting from *b* and b' not only violate the TVC as seen above but also are dominated by the suboptimal path *be* and $b'e'$. That is, by choosing the suboptimal path be such that starting from *b* and once reaches *e* in a finite time $t = T < +\infty$, stays there thereafter, the value of the value function $W = \int_0^\infty u(c)e^{-\rho t}dt$ derived from this suboptimal path is clearly higher than the one obtained from the path *beB*. By the same reason, the path $b'e'\,d\,k_M$ is dominated by the path $b'e'$.

Then we observe that such paths as starting points *b* and b' are never chosen, and that only such paths starting points as *a* and a' are chosen, but these latter paths violate NPG (and hence TVC). In short there exists no optimal path for Case 2-1.

Next we consider

Case 2-2 (Fig. 10.2)

This case is similar to Case 2-1. Such paths as *beB* and $b'e'\,d\,k_M$ are dominated by respectively the suboptimal paths *be* and $b'e'$. Hence only paths starting from *a* and a' should be considered. Both paths converge to points on the horizontal axis in a finite time $t = T < -\infty$, implying $c = 0$ at $t = T < +\infty$, violating TVC, $H = 0$ at $t = T$ (11) in view of (5) and (6) as discussed before.

In short there exists no optimal path for Case 2-2.

Lastly we discuss

Case 2-3 (Fig. 10.3)

As discussed in Case 2-1 and 2-2, such paths as starting from b and b' are not optimal. Furthermore also such paths as starting from points a and a' and converging to some points on the horizontal axis are not optimal since these are dominated respectively by the suboptimal paths aE and $a'E'$ as discussed above.

In short there exists no optimal path for Case 2-3. Then we can obtain that there exists no optimal path for Case 2, i.e., $\bar{k} \leq \underline{k}$ $(\rho - r_0 + \geq e^{\beta/\alpha})$.

IV. Concluding Remarks

There exists several common characteristics of the results of II $(r(b))$ and III $(r(b/k))$. As seen from Case 1 of both sections, the stationary state E exists if positive consumption $c = \bar{c} = f(\bar{k}) - g(\bar{k})$ is realized at E where $k = \bar{k}$ is defined by $\rho = f'(k)$. Sufficient conditions for this to occur are (1) time discount rate ρ is not so large, (2) the foreign debt $b = \bar{b}$ at E is not so large where $\bar{b}$ is defined by $f'(\bar{k}) = r(\bar{b})$ or $f'(\bar{k}) = r(\bar{b}/\bar{k})$, which is possible if the rate of increase in r is not so high – i.e., the home country's risk premium does not increase so rapidly and (3) the level of productivity is not so low.[12/]

In other words if some of these conditions fail to be met, then the non-existence of the stationary state, and hence of the optimal path may occur. Especially if we recall that in some developing countries the time discount rates are high reflecting low level of consumption, risk premium is rather high ??? to accumulated foreign debt and level of labor productivity is low as a result of low rates of economic growth or of poor access to advanced foreign technology, then the plausibility of such a non-existence seems not negligible. Furthermore, even if the stationary state exists (i.e., Case 1 of II and III), if the initial level of per capita capital k_0 is less than $\underline{k}$, then the economy cannot reach the stationary state for $r(b)$ (i.e., II) and for $r(b/k)$ (i.e., III) with $k^* < \underline{k}$. Again if we recall the capital scarcity of the developing countries, this possibility cannot be denied easily. The most significant difference between II ($r(b)$ and III $(r(b/k))$ lies in the possibility of the occurrence of the optimal path toward the stationary state. In the former if the initial capital stock k_0 is less than $\underline{k}$, then the economy never reach the optimal path. However in the latter even if k_0 is less than $\underline{k}$, if k^* is larger than $\underline{k}$ when k^* is the value of k at which the directions of k and a are reversed, then the economy can reach the optimal path as shown in Case 1-2 and 1-3 in III.

Furthermore, another difference between these are the possibility of the discontinuity of the optimal paths of Cases 1-2 and 1-3 in III (i.e., $r(b/k)$), and continuity of the optimal path in II (i.e., $(r(b))$. To our best knowledge this result is new, we admit our model to be very simple – no government expenditure, no public investment, fixed time discount rate, and no investment adjustment costs etc., which are left for our net exercise. Also one natural way for generalization would be to extend into two – country model.

Figures

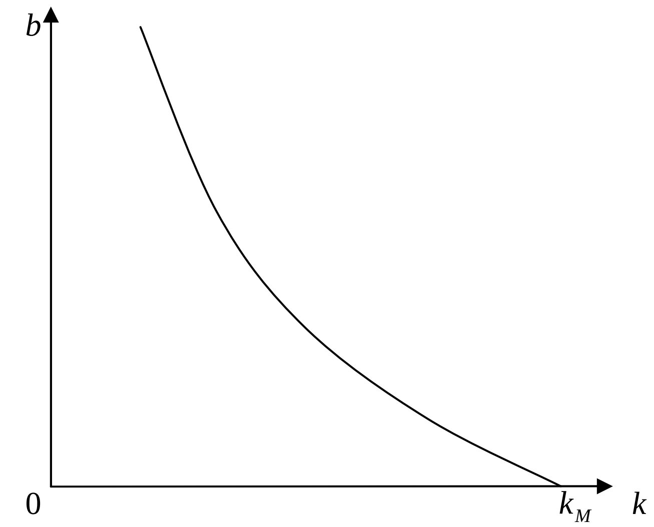

Fig. 1

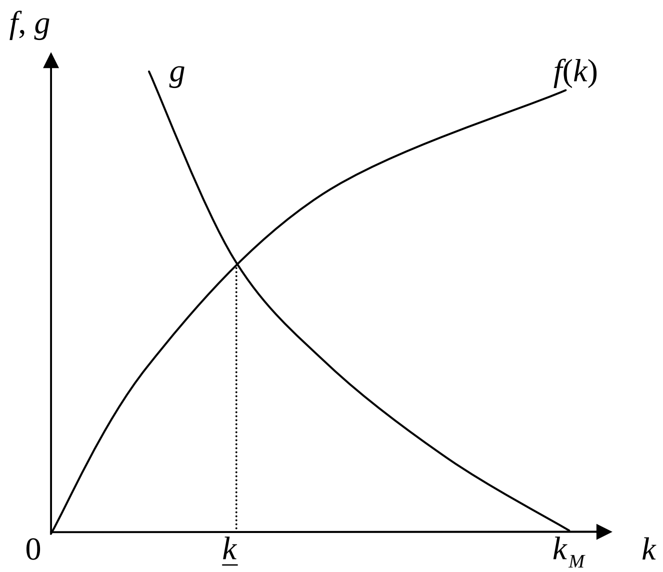

Fig. 2

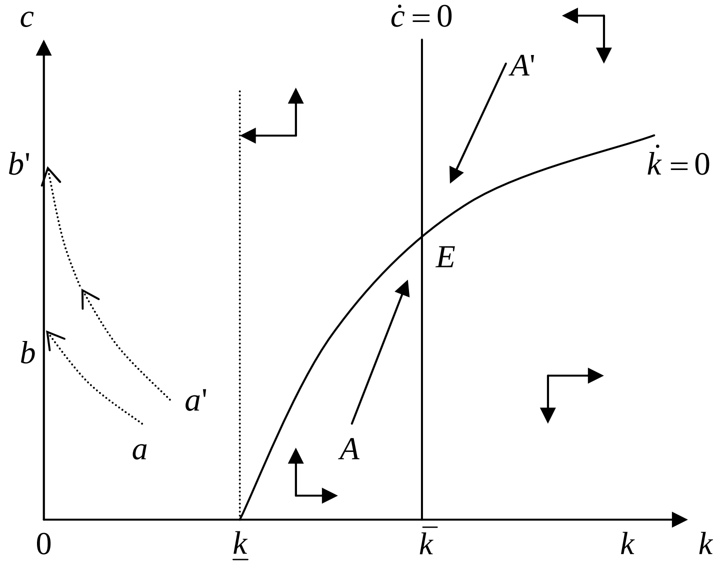

Fig. 3 (Case 1)

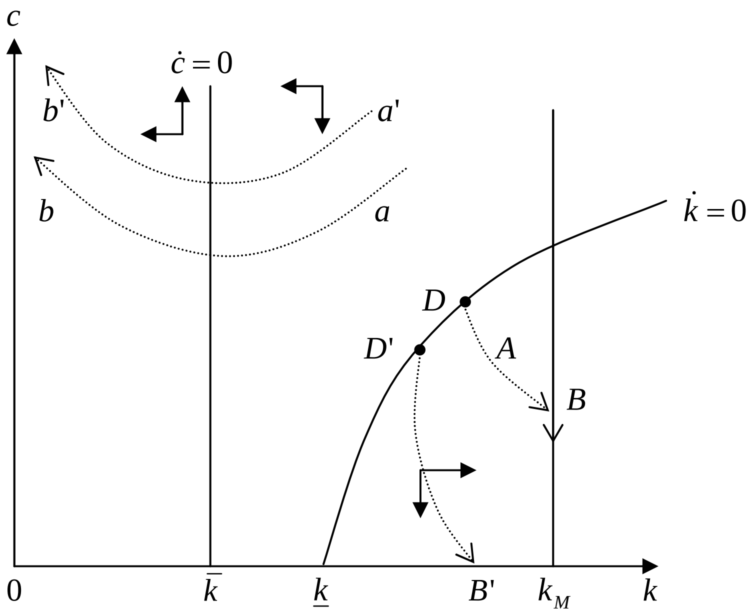

Fig. 4 (Case 2)

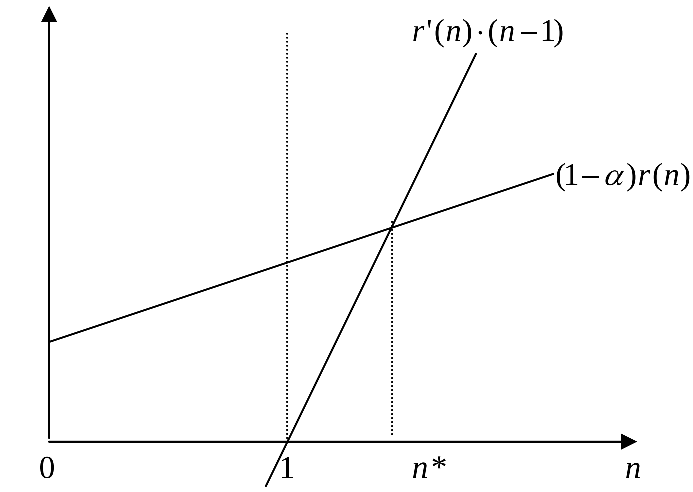

Fig. 5

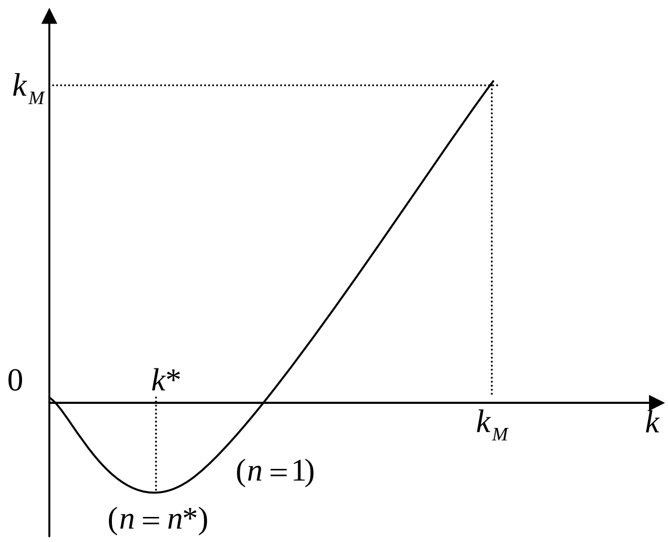

Fig. 6

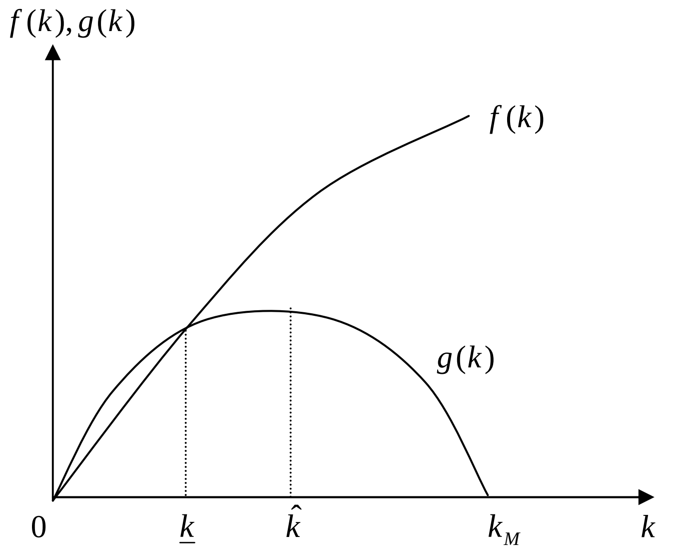

Fig. 7

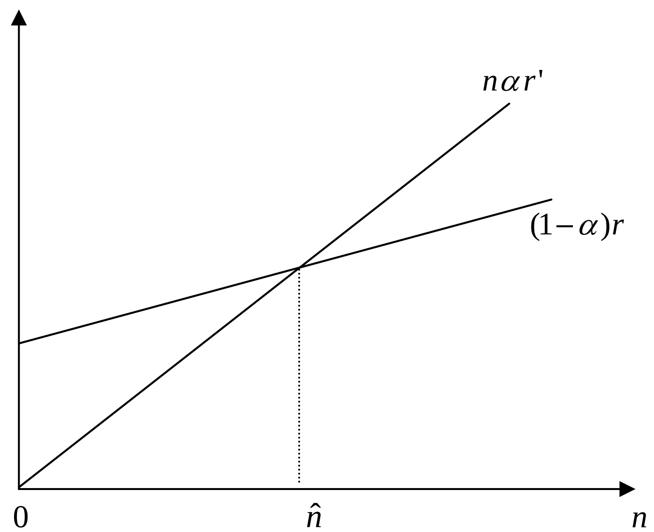

Fig. 8

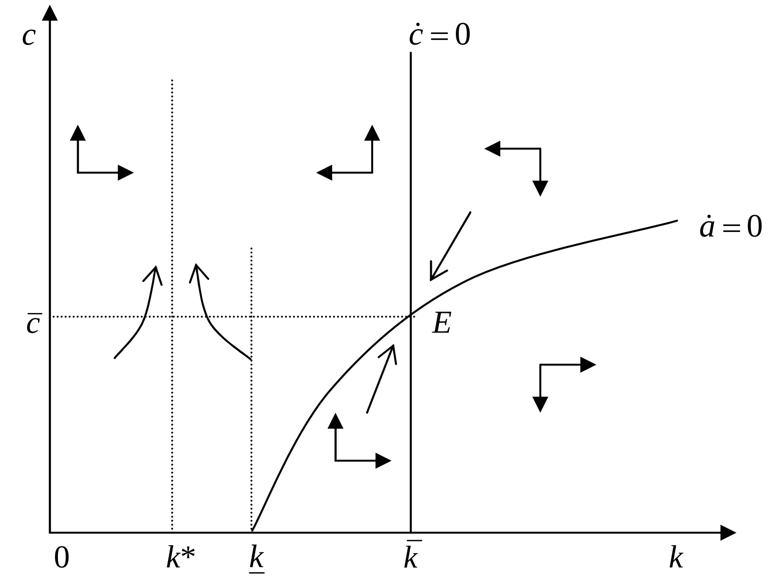

Fig. 9.1(Case1-1)

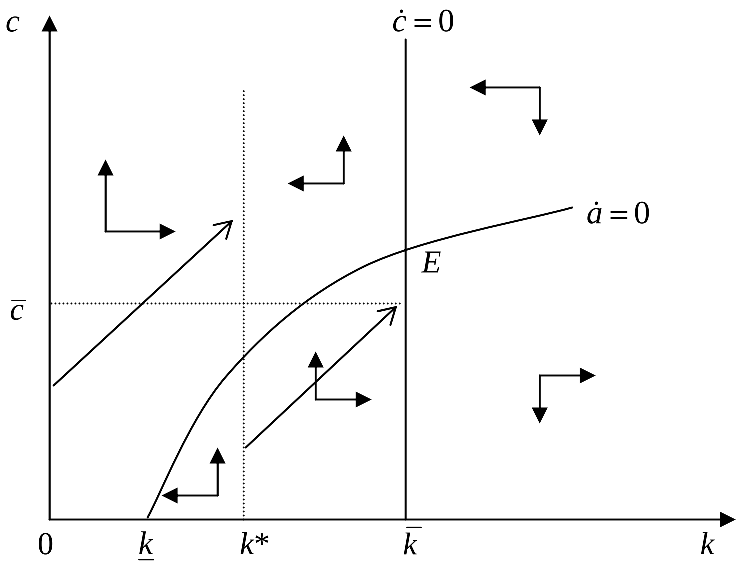

Fig.9.2(Case1-2)

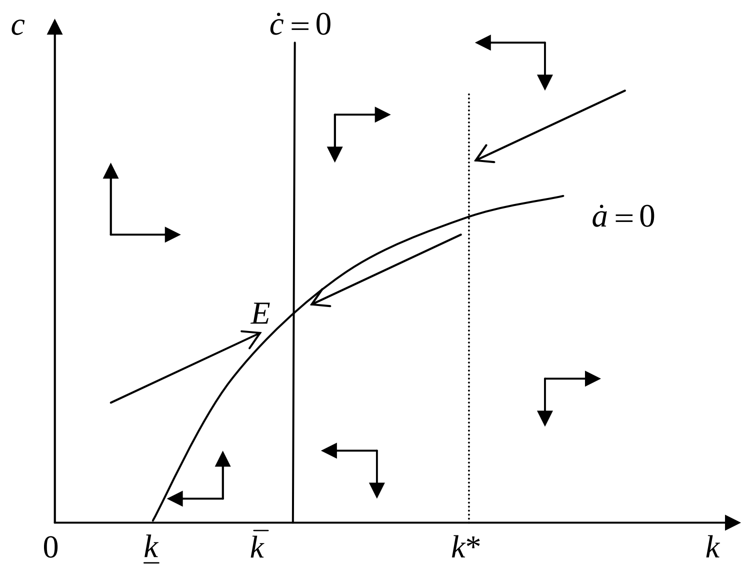

Fig.9.3(Case1-3)

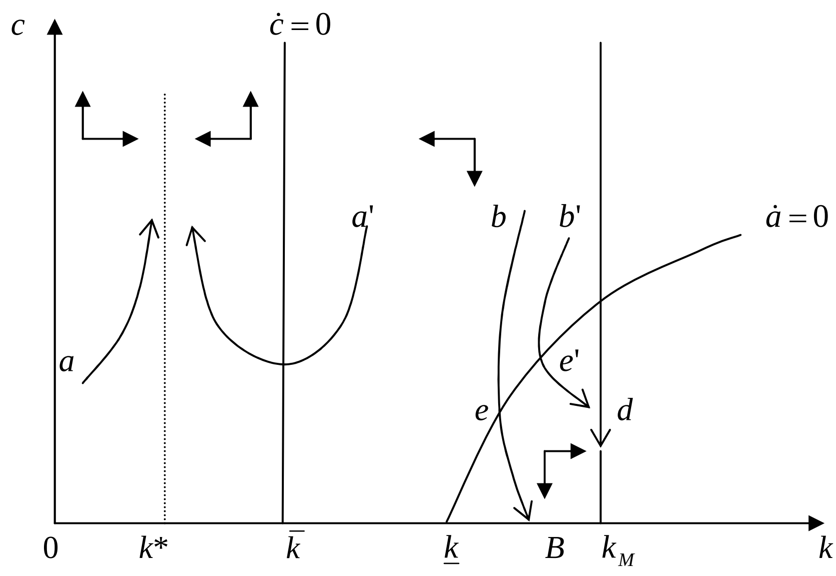

Fig.10.1(Case2-1)

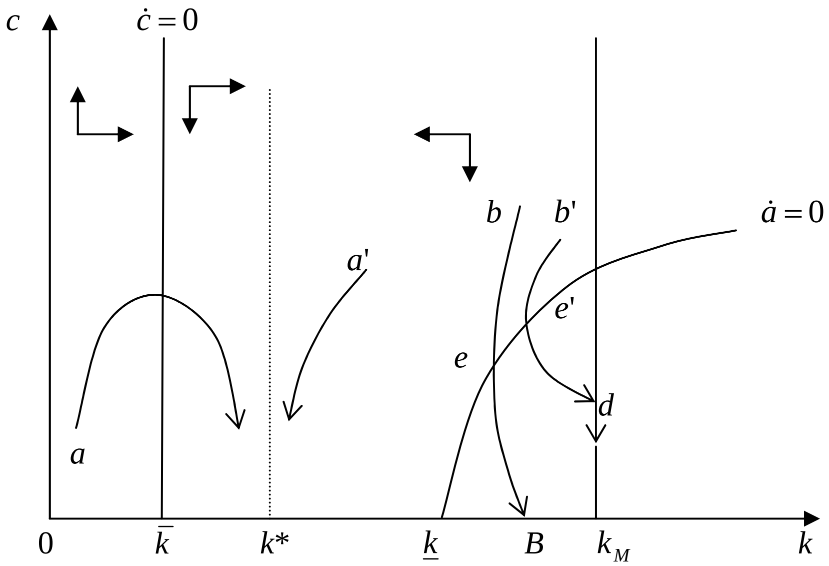

Fig.10.2(Case2-2)

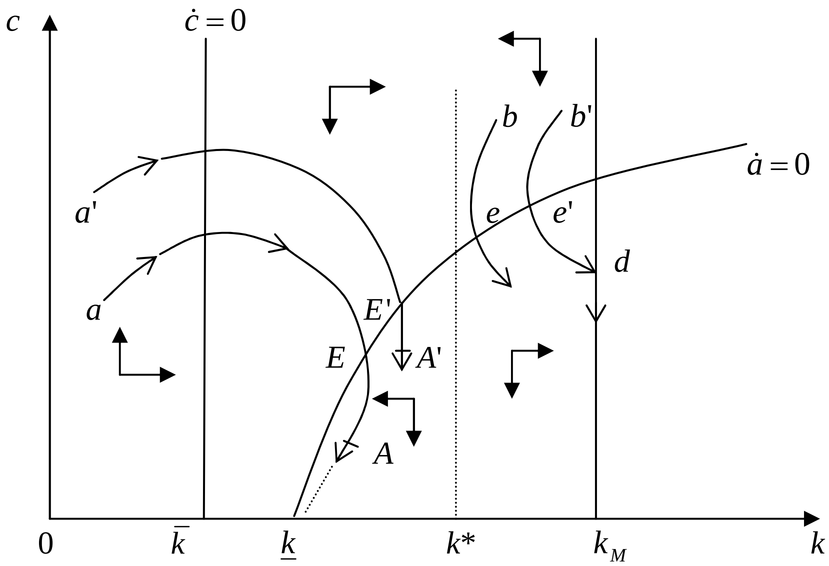

Fig.10.3(Case2-3)

Footnotes

1. Karayalcin(1994) also employs the adjustment costs of investment.
2. The following arguments go through also with $u(c)=\log c$ when $\sigma=1$
3. See, e.g., Leonard and Vang Long(1992), 7.5.
4. From (8), we obtain $\lambda(t)=\lambda(0)e^{-\int_0^t(r(\tau)-\rho)d\tau}$. Substituting this into (9), we obtain $\lim_{t\to\infty} ae^{-\int_0^t r(\tau)d\tau}=\lim_{t\to\infty} ke^{-\int_0^t r(\tau)d\tau}-\lim_{t\to\infty} be^{-\int_0^t r(\tau)d\tau}=0$. Here noting that $\lim_{t\to\infty} ke^{-\int_0^t r(\tau)d\tau}=0 \Leftrightarrow \dot{k}/k-\gamma<0$ as $t\to\infty$, and from $r\to+\infty$ as $k\to 0$, and $\dot{k}/k\to 0$ from $k\to 0$ as $t\to\infty$, we obtain $\lim_{t\to\infty} ke^{-\int_0^t r(\tau)d\tau}=0$. Then (9) is reduced to (11).
5. NPG is equivalent to $\dot{b}/b-r<0$ as $t\to\infty$. From (2) and (3) $\dot{b}/b-r<0 \Leftrightarrow -ex/b<0 \Leftrightarrow \dfrac{i+c-f(k)}{b}<0$ as $t\to\infty$. Recalling $i\to 0$, $k\to 0$ and $c\to\infty$ along $a'b'$ as $t\to\infty$, $i+c-f(k)>0$ for sufficiently lays *t*, implying NPG is violated.
6. We follow this specification to Chatterjee and Trunovsky (2005).
7. From (7), (14) and $f(k)=k^{\alpha}$, we obtain $\alpha k^{\alpha-1}=r_0+e^{\beta n}-1$ when $n=b/k=1-a/k$. This is rewritten as $a=k(1-\dfrac{1}{\beta}\log(\alpha k^{\alpha-1}+1-r_0))$. Hence it suffices to show $\lim_{k\to 0}\dfrac{\log(\alpha k^{\alpha-1}+1-r_o)}{\frac{1}{k}}=\lim_{k\to 0}\dfrac{-k^2\cdot\alpha(\alpha-1)k^{\alpha-2}}{\alpha k^{\alpha-1}+1-r_0}=\lim_{k\to 0}\dfrac{\alpha(1-\alpha)k^{\alpha}}{\alpha k^{\alpha-1}+1-r_0}=0$ in order to derive $a\to 0$ as $k\to 0$.
8. From (7), (14) and $f(k)=k^{\alpha}$, we obtain $0\le g(k)=r(k-a)=f'(k)(k-a)\le \alpha f(k)$. Since $f(k)\to 0$ as $k\to 0$, we observe $g(k)\to 0$ as $k\to 0$.
9. By definition we observe $g(k)=r\cdot(k-a)=f'(k)(k-a)=f'(k)\cdot\dfrac{k}{\beta}\log(\alpha k^{\alpha-1}+1-r_0)$ (see footnote (7) for its derivation) $=\dfrac{\alpha}{\beta}f(k)\cdot\log(\alpha k^{\alpha-1}+1-r_0)$. Hence $f>g \Leftrightarrow 1>\dfrac{\alpha}{\beta}\log(\alpha k^{\alpha-1}+1-r_0)\Leftrightarrow 1>\alpha n$ (from $\alpha k^{\alpha-1}+1-r_0=f'(k)+1-r_0=r+1-r_0=e^{\beta n}$)
10. More precisely let $t=T$ be the time when (*k*, *c*) reaches at (k^*, c_T) where $c_T=c(k^*)$. Then the objective function is expressed as $\int_0^T u(c)e^{-\rho t}dt+e^{-\rho T}\int_0^{\infty}u(c)e^{-\rho t}dt$. Here the both integrals are subject to (4). Given $t=T$, the

objective function is maximized as a function of *T*, *W*(*T*). Then *T* which maximized *W*(*T*) is chosen and hence the optimal path (*k*, *c*) is also chosen.

11. As seen from Fig. 6, $\underline{k} < k^* \Leftrightarrow$ at $k = \underline{k}$ (i.e., $n = 1/\alpha$)
$a'(k) < 0 \Leftrightarrow (1-\alpha)r + r'(1-n) < 0$ at $n = 1/\alpha \Leftrightarrow \alpha r - r' < 0$ at $n = 1/\alpha$
$\Leftrightarrow \alpha(r_0 - 1 + e^{\beta/\alpha}) - \beta e^{\beta/\alpha} < 0 \Leftrightarrow (\alpha - \beta)e^{\beta/\alpha} < \alpha(1 - r_0)$

12. Let $f(k, \underset{+}{\theta})$ and $f'(k, \underset{+}{\theta})$ where $\theta > 0$ being the technological parameter indicating the increase in the level of labor productivity. Then clearly increase in θ increases both $\bar{k}$ and $\bar{c}$ from $\rho = f'(k, \theta)$ and $\bar{c} = f(\bar{k}, \theta) - g(\bar{k})$.

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