CHAPTER

SIX

COMPARATIVE STATICS AND THE CONCEPT OF DERIVATIVE

The present and the two following chapters will be devoted to the methods of comparative-static analysis.

6.1 THE NATURE OF COMPARATIVE STATICS

Comparative statics, as the name suggests, is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogenous variables. For purposes of such a comparison, we always start by assuming a given initial equilibrium state. In the isolated-market model, for example, such an initial equilibrium will be represented by a determinate price $\bar{P}$ and a corresponding quantity $\bar{Q}$. Similarly, in the simple national-income model of (3.23), the initial equilibrium will be specified by a determinate $\bar{Y}$ and a corresponding $\bar{C}$. Now if we let a disequilibrating change occur in the model—in the form of a variation in the value of some parameter or exogenous variable—the initial equilibrium will, of course, be upset. As a result, the various endogenous variables must undergo certain adjustments. If it is assumed that a new equilibrium state relevant to the new values of the data can be defined and attained, the question posed in the comparative-static analysis is: How would the new equilibrium compare with the old?

It should be noted that in comparative statics we again disregard the process of adjustment of the variables; we merely compare the initial (*pre*change) equilibrium state with the final (*post*change) equilibrium state. Also, we again preclude the possibility of instability of equilibrium, for we assume the new equilibrium to be attainable, just as we do for the old.

A comparative-static analysis can be either qualitative or quantitative in nature. If we are interested only in the question of, say, whether an increase in investment I_0 will increase or decrease the equilibrium income $\bar{Y}$, the analysis will be qualitative because the *direction* of change is the only matter considered. But if we are concerned with the *magnitude* of the change in $\bar{Y}$ resulting from a given change in I_0 (that is, the size of the investment multiplier), the analysis will obviously be quantitative. By obtaining a quantitative answer, however, we can automatically tell the direction of change from its algebraic sign. Hence the quantitative analysis always embraces the qualitative.

It should be clear that the problem under consideration is essentially one of finding a *rate of change*: the rate of change of the equilibrium value of an endogenous variable with respect to the change in a particular parameter or exogenous variable. For this reason, the mathematical concept of *derivative* takes on preponderant significance in comparative statics, because that concept—the most fundamental one in the branch of mathematics known as *differential calculus* — is directly concerned with the notion of rate of change! Later on, moreover, we shall find the concept of derivative to be of extreme importance for optimization problems as well.

6.2 RATE OF CHANGE AND THE DERIVATIVE

Even though our present context is concerned only with the rates of change of the equilibrium values of the variables in a model, we may carry on the discussion in a more general manner by considering the rate of change of any variable y in response to a change in another variable x, where the two variables are related to each other by the function

$$y = f(x)$$

Applied in the comparative-static context, the variable y will represent the equilibrium value of an endogenous variable, and x will be some parameter. Note that, for a start, we are restricting ourselves to the simple case where there is only a single parameter or exogenous variable in the model. Once we have mastered this simplified case, however, the extension to the case of more parameters will prove relatively easy.

The Difference Quotient

Since the notion of "change" figures prominently in the present context, a special symbol is needed to represent it. When the variable x changes from the value x_0 to a new value x_1, the change is measured by the difference $x_1 - x_0$. Hence, using the symbol Δ (the Greek capital delta, for "difference") to denote the change, we write $\Delta x = x_1 - x_0$. Also needed is a way of denoting the value of the function $f(x)$ at various values of x. The standard practice is to use the notation $f(x_i)$ to represent the value of $f(x)$ when $x = x_i$. Thus, for the function $f(x) = 5 + x^2$, we have $f(0) = 5 + 0^2 = 5$; and similarly, $f(2) = 5 + 2^2 = 9$, etc.

When x changes from an initial value x_0 to a new value $(x_0 + \Delta x)$, the value of the function $y = f(x)$ changes from $f(x_0)$ to $f(x_0 + \Delta x)$. The change in y per unit of change in x can be represented by the *difference quotient*

$$\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \tag{6.1}$$

This quotient, which measures the average rate of change of y, can be calculated if we know the initial value of x, or x_0, and the magnitude of change in x, or Δx. That is, $\Delta y/\Delta x$ is a function of x_0 and Δx.

Example 1 Given $y = f(x) = 3x^2 - 4$, we can write:

$$f(x_0) = 3(x_0)^2 - 4 \qquad f(x_0 + \Delta x) = 3(x_0 + \Delta x)^2 - 4$$

Therefore, the difference quotient is

$$\frac{\Delta y}{\Delta x} = \frac{3(x_0 + \Delta x)^2 - 4 - (3x_0^2 - 4)}{\Delta x} = \frac{6x_0\Delta x + 3(\Delta x)^2}{\Delta x} = 6x_0 + 3\Delta x \tag{6.2}$$

which can be evaluated if we are given x_0 and Δx. Let $x_0 = 3$ and $\Delta x = 4$; then the average rate of change of y will be $6(3) + 3(4) = 30$. This means that, on the average, as x changes from 3 to 7, the change in y is 30 units per unit change in x.

The Derivative

Frequently, we are interested in the rate of change of y when Δx is very small. In such a case, it is possible to obtain an approximation of $\Delta y/\Delta x$ by dropping all the terms in the difference quotient involving the expression Δx. In (6.2), for instance, if Δx is very small, we may simply take the term $6x_0$ on the right as an approximation of $\Delta y/\Delta x$. The smaller the value of Δx, of course, the closer is the approximation to the true value of $\Delta y/\Delta x$.

As Δx approaches zero (meaning that it gets closer and closer to, but never actually reaches, zero), $(6x_0 + 3\Delta x)$ will approach the value $6x_0$, and by the same token, $\Delta y/\Delta x$ will approach $6x_0$ also. Symbolically, this fact is expressed either by the statement $\Delta y/\Delta x \to 6x_0$ as $\Delta x \to 0$, or by the equation

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (6x_0 + 3\Delta x) = 6x_0 \tag{6.3}$$

where the symbol $\lim\limits_{\Delta x \to 0}$ is read: "The limit of... as Δx approaches 0." If, as $\Delta x \to 0$, the limit of the difference quotient $\Delta y/\Delta x$ exists, that limit is identified as the derivative of the function $y = f(x)$.

Several points should be noted about the derivative. First, a derivative is a *function*; in fact, in this usage the word *derivative* really means a derived function. The original function $y = f(x)$ is a *primitive function*, and the derivative is another function derived from it. Whereas the difference quotient is a function of x_0 and Δx, you should observe—from (6.3), for instance—that the derivative is a function of x_0 only. This is because Δx is already compelled to approach zero, and therefore it should not be regarded as another variable in the function. Let us also add that so far we have used the subscripted symbol x_0 only in order to stress the fact that a change in x must start from some specific value of x. Now that this is understood, we may delete the subscript and simply state that the derivative, like the primitive function, is itself a function of the independent variable x. That is, for each value of x, there is a unique corresponding value for the derivative function.

Second, since the derivative is merely a limit of the difference quotient, which measures a rate of change of y, the derivative must of necessity also be a measure of some rate of change. In view of the fact that the change in x envisaged in the derivative concept is infinitesimal (that is, $\Delta x \to 0$), however, the rate measured by the derivative is in the nature of an *instantaneous* rate of change.

Third, there is the matter of notation. Derivative functions are commonly denoted in two ways. Given a primitive function $y = f(x)$, one way of denoting its derivative (if it exists) is to use the symbol $f'(x)$, or simply f'; this notation is attributed to the mathematician Lagrange. The other common notation is dy/dx, devised by the mathematician Leibniz. [Actually there is a third notation, Dy, or $Df(x)$, but we shall not use it in the following discussion.] The notation $f'(x)$, which resembles the notation for the primitive function $f(x)$, has the advantage of conveying the idea that the derivative is itself a function of x. The reason for expressing it as $f'(x)$—rather than, say, $\phi(x)$—is to emphasize that the function f' is derived from the primitive function f. The alternative notation, dy/dx, serves instead to emphasize that the value of a derivative measures a rate of change. The letter d is the counterpart of the Greek Δ, and dy/dx differs from $\Delta y/\Delta x$ chiefly in that the former is the limit of the latter as Δx approaches zero. In the subsequent discussion, we shall use both of these notations, depending on which seems the more convenient in a particular context.

Using these two notations, we may define the derivative of a given function $y = f(x)$ as follows:

$$\frac{dy}{dx} \equiv f'(x) \equiv \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Example 2 Referring to the function $y = 3x^2 - 4$ again, we have shown its difference quotient to be (6.2), and the limit of that quotient to be (6.3). On the basis of the latter, we may now write (replacing x_0 with x):

$$\frac{dy}{dx} = 6x \qquad \text{or} \qquad f'(x) = 6x$$

Note that different values of x will give the derivative correspondingly different values. For instance, when $x = 3$, we have $f'(x) = 6(3) = 18$; but when $x = 4$, we find that $f'(4) = 6(4) = 24$.

EXERCISE 6.2

1 Given the function $y = 4x^2 + 9$:
(*a*) Find the difference quotient as a function of x and Δx. (Use x in lieu of x_0).
(*b*) Find the derivative dy/dx.
(*c*) Find $f'(3)$ and $f'(4)$.

2 Given the function $y = 5x^2 - 4x$:
(*a*) Find the difference quotient as a function of x and Δx.
(*b*) Find the derivative dy/dx.
(*c*) Find $f'(2)$ and $f'(3)$.

3 Given the function $y = 5x - 2$:
(*a*) Find the difference quotient $\Delta y/\Delta x$. What type of function is it?
(*b*) Since the expression Δx does not appear in the function $\Delta y/\Delta x$ above, does it make any difference to the value of $\Delta y/\Delta x$ whether Δx is large or small? Consequently, what is the limit of the difference quotient as Δx approaches zero?

6.3 THE DERIVATIVE AND THE SLOPE OF A CURVE

Elementary economics tells us that, given a total-cost function $C = f(Q)$, where C denotes total cost and Q the output, the marginal cost (MC) is defined as the change in total cost resulting from a unit increase in output; that is, $\text{MC} = \Delta C/\Delta Q$. It is understood that ΔQ is an extremely small change. For the case of a product that has discrete units (integers only), a change of one unit is the smallest change possible; but for the case of a product whose quantity is a continuous variable, ΔQ will refer to an infinitesimal change. In this latter case, it is well known that the marginal cost can be measured by the slope of the total-cost curve. But the slope of the total-cost curve is nothing but the limit of the ratio $\Delta C/\Delta Q$, when ΔQ approaches zero. Thus the concept of the slope of a curve is merely the geometric counterpart of the concept of the derivative. Both have to do with the "marginal" notion so extensively used in economics.

In Fig. 6.1, we have drawn a total-cost curve C, which is the graph of the (primitive) function $C = f(Q)$. Suppose that we consider Q_0 as the initial output level from which an increase in output is measured, then the relevant point on the cost curve will be A. If output is to be raised to $Q_0 + \Delta Q = Q_2$, the total cost will be increased from C_0 to $C_0 + \Delta C = C_2$; thus $\Delta C/\Delta Q = (C_2 - C_0)/(Q_2 - Q_0)$. Geometrically, this is the ratio of two line segments, EB/AE, or the *slope* of the line AB. This particular ratio measures an average rate of change—the *average*

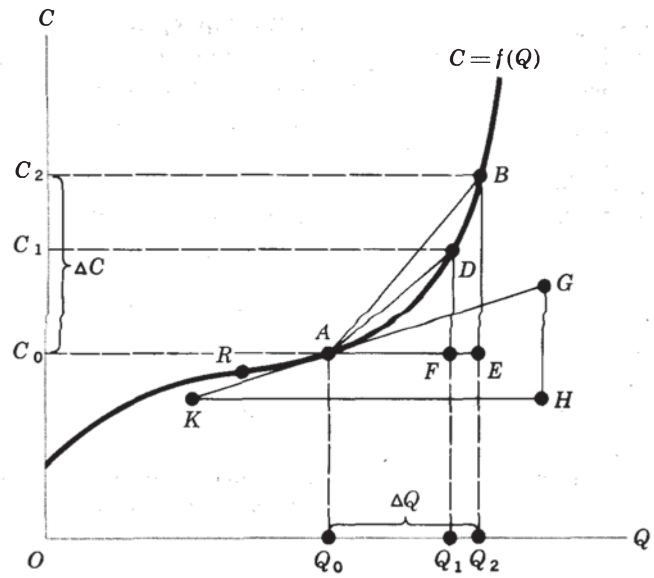

Figure 6.1

marginal cost for the particular ΔQ pictured—and represents a difference quotient. As such, it is a function of the initial value Q_0 and the amount of change ΔQ.

What happens when we vary the magnitude of ΔQ? If a smaller output increment is contemplated (say, from Q_0 to Q_1 only), then the average marginal cost will be measured by the slope of the line AD instead. Moreover, as we reduce the output increment further and further, flatter and flatter lines will result until, in the limit (as $\Delta Q \to 0$), we obtain the line KG (which is the *tangent line* to the cost curve at point A) as the relevant line. The slope of $KG(= HG/KH)$ measures the slope of the total-cost curve at point A and represents the limit of $\Delta C/\Delta Q$, as $\Delta Q \to 0$, when initial output is at $Q = Q_0$. Therefore, in terms of the derivative, the slope of the $C = f(Q)$ curve at point A corresponds to the particular derivative value $f'(Q_0)$.

What if the initial output level is changed from Q_0 to, say, Q_2? In that case, point B on the curve will replace point A as the relevant point, and the slope of the curve at the new point B will give us the derivative value $f'(Q_2)$. Analogous results are obtainable for alternative initial output levels. In general, the derivative $f'(Q)$—a function of Q—will vary as Q changes.

6.4 THE CONCEPT OF LIMIT

The derivative dy/dx has been defined as the limit of the difference quotient $\Delta y/\Delta x$ as $\Delta x \to 0$. If we adopt the shorthand symbols $q \equiv \Delta y/\Delta x$ (q for quotient) and $v \equiv \Delta x$ (v for variation), we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{v \to 0} q$$

In view of the fact that the derivative concept relies heavily on the notion of limit, it is imperative that we get a clear idea about that notion.

Left-Side Limit and Right-Side Limit

The concept of limit is concerned with the question: "What value does one variable (say, q) approach as another variable (say, v) approaches a specific value (say, zero)?" In order for this question to make sense, q must, of course, be a function of v; say, $q = g(v)$. Our immediate interest is in finding the limit of q as $v \to 0$, but we may just as easily explore the more general case of $v \to N$, where N is any finite real number. Then, $\lim_{v \to 0} q$ will be merely a special case of $\lim_{v \to N} q$ where $N = 0$. In the course of the discussion, we shall actually also consider the limit of q as $v \to +\infty$ (plus infinity) or as $v \to -\infty$ (minus infinity).

When we say $v \to N$, the variable v can approach the number N either from values greater than N, or from values less than N. If, as $v \to N$ from the left side (from values less than N), q approaches a finite number L, we call L the *left-side limit* of q. On the other hand, if L is the number that q tends to as $v \to N$ from the right side (from values greater than N), we call L the *right-side limit* of q. The left- and right-side limits may or may not be equal.

The left-side limit of q is symbolized by $\lim_{v \to N^-} q$ (the minus sign signifies from values less than N), and the right-side limit is written as $\lim_{v \to N^+} q$. When—and only when—the two limits have a common finite value (say, L), we consider the limit of q to exist and write it as $\lim_{v \to N} q = L$. Note that L must be a *finite* number. If we have the situation of $\lim_{v \to N} q = \infty$ (or $-\infty$), we shall consider q to possess *no* limit, because $\lim_{v \to N} q = \infty$ means that $q \to \infty$ as $v \to N$, and if q will assume *ever-increasing* values as v tends to N, it would be contradictory to say that q has a limit. As a convenient way of expressing the fact that $q \to \infty$ as $v \to N$, however, people do indeed write $\lim_{v \to N} q = \infty$ and speak of q as having an "infinite limit."

In certain cases, only the limit of one side needs to be considered. In taking the limit of q as $v \to +\infty$, for instance, only the left-side limit of q is relevant, because v can approach $+\infty$ only from the left. Similarly, for the case of $v \to -\infty$, only the right-side limit is relevant. Whether the limit of q exists in these cases will depend only on whether q approaches a finite value as $v \to +\infty$, or as $v \to -\infty$.

It is important to realize that the symbol ∞ (infinity) is not a number, and therefore it cannot be subjected to the usual algebraic operations. We cannot have

$3 + \infty$ or $1/\infty$; nor can we write $q = \infty$, which is not the same as $q \to \infty$. However, it is acceptable to express the *limit* of q as "$=$" (as against $\to$) ∞, for this merely indicates that $q \to \infty$.

Graphical Illustrations

Let us illustrate, in Fig. 6.2, several possible situations regarding the limit of a function $q = g(v)$.

Figure 6.2*a* shows a smooth curve. As the variable v tends to the value N from *either* side on the horizontal axis, the variable q tends to the value L. In this case, the left-side limit is identical with the right-side limit; therefore we can write $\lim_{v \to N} q = L$.

The curve drawn in Fig. 6.2*b* is not smooth; it has a sharp turning point directly above the point N. Nevertheless, as v tends to N from either side, q again tends to an identical value L. The limit of q again exists and is equal to L.

Figure 6.2*c* shows what is known as a *step function*.* In this case, as v tends to N, the left-side limit of q is L_1, but the right-side limit is L_2, a different number. Hence, q does not have a limit as $v \to N$.

Lastly, in Fig. 6.2*d*, as v tends to N, the left-side limit of q is $-\infty$, whereas the right-side limit is $+\infty$, because the two parts of the (hyperbolic) curve will fall and rise indefinitely while approaching the broken vertical line as an asymptote. Again, $\lim_{v \to N} q$ does not exist. On the other hand, if we are considering a different sort of limit in diagram *d*, namely, $\lim_{v \to +\infty} q$, then only the left-side limit has relevance, and we do find that limit to exist: $\lim_{v \to +\infty} q = M$. Analogously, you can verify that $\lim_{v \to -\infty} q = M$ as well.

It is also possible to apply the concepts of left-side and right-side limits to the discussion of the marginal cost in Fig. 6.1. In that context, the variables q and v will refer, respectively, to the quotient $\Delta C/\Delta Q$ and to the magnitude of ΔQ, with all changes being measured from point A on the curve. In other words, q will refer to the slope of such lines as AB, AD, and KG, whereas v will refer to the length of such lines as Q_0Q_2 (= line AE) and Q_0Q_1 (= line AF). We have already seen that, as v approaches zero from a positive value, q will approach a value equal to the slope of line KG. Similarly, we can establish that, if ΔQ approaches zero from

* This name is easily explained by the shape of the curve. But step functions can be expressed algebraically, too. The one illustrated in Fig. 6.2*c* can be expressed by the equation

$$q = \begin{cases} L_1 & (\text{for } 0 \le v < N) \\ L_2 & (\text{for } N \le v) \end{cases}$$

Note that, in each subset of its domain described above, the function appears as a distinct constant function, which constitutes a "step" in the graph.

In economics, step functions can be used, for instance, to show the various prices charged for different quantities purchased (the curve shown in Fig. 6.2*c* pictures ***quantity discount***) or the various tax rates applicable to different income brackets.

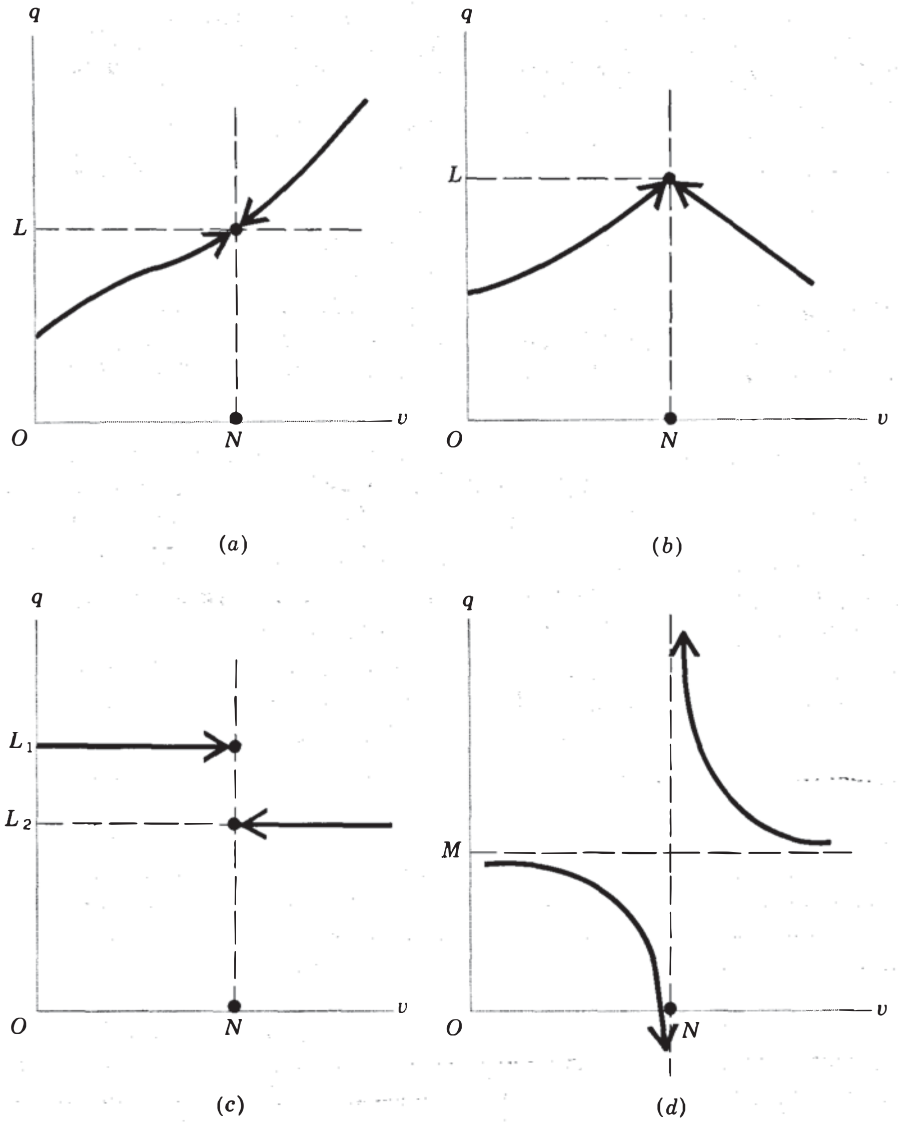

Figure 6.2

a negative value (i.e., as the *decrease* in output becomes less and less), the quotient $\Delta C/\Delta Q$, as measured by the slope of such lines as RA (not drawn), will also approach a value equal to the slope of line KG. Indeed, the situation here is very much akin to that illustrated in Fig. 6.2*a*. Thus the slope of KG in Fig. 6.1 (the counterpart of L in Fig. 6.2) is indeed the limit of the quotient q as v tends to zero, and as such it gives us the marginal cost at the output level $Q = Q_0$.

Evaluation of a Limit

Let us now illustrate the algebraic evaluation of a limit of a given function $q = g(v)$.

Example 1 Given $q = 2 + v^2$, find $\lim_{v \to 0} q$. To take the left-side limit, we substitute the series of negative values $-1, -\frac{1}{10}, -\frac{1}{100}, \ldots$ (in that order) for v and find that $(2 + v^2)$ will decrease steadily and approach 2 (because v^2 will gradually approach 0). Next, for the right-side limit, we substitute the series of positive values $1, \frac{1}{10}, \frac{1}{100}, \ldots$ (in that order) for v and find the same limit as before. Inasmuch as the two limits are identical, we consider the limit of q to exist and write $\lim_{v \to 0} q = 2$.

It is tempting to regard the answer just obtained as the outcome of setting $v = 0$ in the equation $q = 2 + v^2$, but this temptation should in general be resisted. In evaluating $\lim_{v \to N} q$, we only let v *tend to* N but, as a rule, do not let $v = N$. Indeed, we can quite legitimately speak of the limit of q as $v \to N$, even if N is *not* in the domain of the function $q = g(v)$. In this latter case, if we try to set $v = N$, q will clearly be undefined.

Example 2 Given $q = (1 - v^2)/(1 - v)$, find $\lim_{v \to 1} q$. Here, $N = 1$ is not in the domain of the function, and we cannot set $v = 1$ because that would involve division by zero. Moreover, even the limit-evaluation procedure of letting $v \to 1$, as used in Example 1, will cause difficulty, for the denominator $(1 - v)$ will approach zero when $v \to 1$, and we will still have no way of performing the division in the limit.

One way out of this difficulty is to try to transform the given ratio to a form in which v will not appear in the denominator. Since $v \to 1$ implies that $v \neq 1$, so that $(1 - v)$ is nonzero, it is legitimate to divide the expression $(1 - v^2)$ by $(1 - v)$, and write*

$$q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1)$$

* The division can be performed, as in the case of numbers, in the following manner:

$$\begin{array}{r|l} & 1 + v \\ \hline 1 - v & 1 \quad - v^2 \\ & 1 - v \\ \hline & \quad v - v^2 \\ & \quad \underline{\underline{v - v^2}} \end{array}$$

Alternatively, we may resort to factoring as follows:

$$\frac{1 - v^2}{1 - v} = \frac{(1 + v)(1 - v)}{1 - v} = 1 + v \qquad (v \neq 1)$$

In this new expression for q, there is no longer a denominator with v in it. Since $(1 + v) \to 2$ as $v \to 1$ from *either* side, we may then conclude that $\lim_{v \to 1} q = 2$.

Example 3 Given $q = (2v + 5)/(v + 1)$, find $\lim_{v \to +\infty} q$. The variable v again appears in *both* the numerator and the denominator. If we let $v \to +\infty$ in both, the result will be a ratio between two infinitely large numbers, which does not have a clear meaning. To get out of the difficulty, we try this time to transform the given ratio to a form in which the variable v will not appear in the numerator.* This, again, can be accomplished by dividing out the given ratio. Since $(2v + 5)$ is not evenly divisible by $(v + 1)$, however, the result will contain a remainder term as follows:

$$q = \frac{2v + 5}{v + 1} = 2 + \frac{3}{v + 1}$$

But, at any rate, this new expression for q no longer has a numerator with v in it. Noting that the remainder $3/(v + 1) \to 0$ as $v \to +\infty$, we can then conclude that $\lim_{v \to +\infty} q = 2$.

There also exist several useful theorems on the evaluation of limits. These will be discussed in Sec. 6.6.

Formal View of the Limit Concept

The above discussion should have conveyed some general ideas about the concept of limit. Let us now give it a more precise definition. Since such a definition will make use of the concept of *neighborhood* of a point on a line (in particular, a specific number as a point on the line of real numbers), we shall first explain the latter term.

For a given number L, there can always be found a number $(L - a_1) < L$ and another number $(L + a_2) > L$, where a_1 and a_2 are some arbitrary positive numbers. The set of all numbers falling between $(L - a_1)$ and $(L + a_2)$ is called the *interval* between those two numbers. If the numbers $(L - a_1)$ and $(L + a_2)$ are included in the set, the set is a *closed interval*; if they are excluded, the set is an *open interval*. A closed interval between $(L - a_1)$ and $(L + a_2)$ is denoted by the bracketed expression

$$[L - a_1, L + a_2] \equiv \{q \mid L - a_1 \le q \le L + a_2\}$$

and the corresponding *open* interval is denoted with parentheses:

$$(L - a_1, L + a_2) \equiv \{q \mid L - a_1 < q < L + a_2\} \tag{6.4}$$

* Note that, unlike the $v \to 0$ case, where we want to take v out of the *denominator* in order to avoid division by zero, the $v \to \infty$ case is better served by taking v out of the *numerator*. As $v \to \infty$, an expression containing v in the numerator will become infinite but an expression with v in the denominator will, more conveniently for us, approach zero and quietly vanish from the scene.

Thus, [ ] relate to the weak inequality sign $\leq$, whereas ( ) relate to the strict inequality sign $<$. But in both types of intervals, the smaller number $(L - a_1)$ is always listed first. Later on, we shall also have occasion to refer to *half-open and half-closed* intervals such as $(3, 5]$ and $[6, \infty)$, which have the following meanings:

$$(3,5] \equiv \{x \mid 3 < x \leq 5\} \qquad [6, \infty) \equiv \{x \mid 6 \leq x < \infty\}$$

Now we may define a *neighborhood* of L to be an open interval as defined in (6.4), which is an interval "covering" the number L.* Depending on the magnitudes of the arbitrary numbers a_1 and a_2, it is possible to construct various neighborhoods for the given number L. Using the concept of neighborhood, the limit of a function may then be defined as follows:

> As v approaches a number N, the limit of $q = g(v)$ is the number L, if, for every neighborhood of L that can be chosen, *however small*, there can be found a corresponding neighborhood of N (excluding the point $v = N$) in the domain of the function such that, for every value of v in that N-neighborhood, its image lies in the chosen L-neighborhood.

This statement can be clarified with the help of Fig. 6.3, which resembles Fig. 6.2a. From what was learned about the latter figure, we know that $\lim_{v \to N} q = L$ in Fig. 6.3. Let us show that L does indeed fulfill the new definition of a limit. As the first step, select an arbitrary small neighborhood of L, say, $(L - a_1,\ L + a_2)$. (This should have been made even smaller, but we are keeping it relatively large to facilitate exposition.) Now construct a neighborhood of N, say, $(N - b_1,\ N + b_2)$, such that the two neighborhoods (when extended into quadrant I) will together define a rectangle (shaded in diagram) with two of its corners lying on the given curve. It can then be verified that, for every value of v in this neighborhood of N (not counting $v = N$), the corresponding value of $q = g(v)$ lies in the chosen neighborhood of L. In fact, no matter how *small* an L-neighborhood we choose, a (correspondingly small) N-neighborhood can be found with the property just cited. Thus L fulfills the definition of a limit, as was to be demonstrated.

We can also apply the above definition to the step function of Fig. 6.2c in order to show that neither L_1 nor L_2 qualifies as $\lim_{v \to N} q$. If we choose a very small neighborhood of L_1—say, just a hair's width on each side of L_1—then, no matter what neighborhood we pick for N, the rectangle associated with the two neighborhoods cannot possibly enclose the lower step of the function. Consequently, for any value of $v > N$, the corresponding value of q (located on the lower step) will not be in the neighborhood of L_1, and thus L_1 fails the test for a limit. By similar reasoning, L_2 must also be dismissed as a candidate for $\lim_{v \to N} q$. In fact, in this case no limit exists for q as $v \to N$.

* The identification of an open interval as the neighborhood of a point is valid only when we are considering a point on a line (one-dimensional space). In the case of a point in a plane (two-dimensional space), its neighborhood must be thought of as an area, say, a circular area around the point.

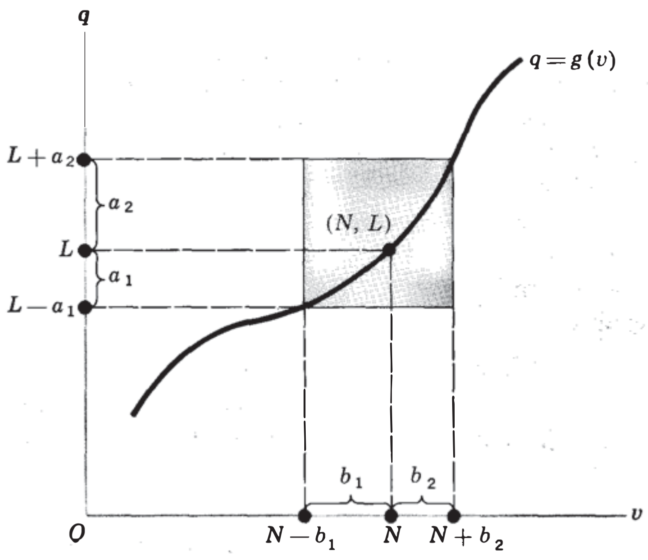

Figure 6.3

The fulfillment of the definition can also be checked algebraically rather than by graph. For instance, consider again the function

$$(6.5) \qquad q = \frac{1 - v^2}{1 - v} = 1 + v \qquad (v \neq 1)$$

It has been found in Example 2 that $\lim_{v \to 1} q = 2$; thus, here we have $N = 1$ and $L = 2$. To verify that $L = 2$ is indeed the limit of q, we must demonstrate that, for every chosen neighborhood of L, $(2 - a_1, 2 + a_2)$, there exists a neighborhood of N, $(1 - b_1, 1 + b_2)$, such that, whenever v is in this neighborhood of N, q must be in the chosen neighborhood of L. This means essentially that, for given values of a_1 and a_2, however small, two numbers b_1 and b_2 must be found such that, whenever the inequality

$$(6.6) \qquad 1 - b_1 < v < 1 + b_2 \qquad (v \neq 1)$$

is satisfied, another inequality of the form

$$(6.7) \qquad 2 - a_1 < q < 2 + a_2$$

must also be satisfied. To find such a pair of numbers b_1 and b_2, let us first rewrite (6.7) by substituting (6.5):

$$(6.7') \qquad 2 - a_1 < 1 + v < 2 + a_2$$

This, in turn, can be transformed into the inequality

$$(6.7'') \qquad 1 - a_1 < v < 1 + a_2$$

A comparison of (6.7″)—a variant of (6.7)—with (6.6) suggests that if we choose the two numbers b_1 and b_2 to be $b_1 = a_1$ and $b_2 = a_2$, the two inequalities (6.6) and (6.7) will always be satisfied simultaneously. Thus the neighborhood of N, $(1 - b_1, 1 + b_2)$, as required in the definition of a limit, can indeed be found for the case of $L = 2$, and this establishes $L = 2$ as the limit.

Let us now utilize the definition of a limit in the opposite way, to show that another value (say, 3) cannot qualify as $\lim_{v \to 1} q$ for the function in (6.5). If 3 were that limit, it would have to be true that, for every chosen neighborhood of 3, $(3 - a_1, 3 + a_2)$, there exists a neighborhood of 1, $(1 - b_1, 1 + b_2)$, such that, whenever v is in the latter neighborhood, q must be in the former neighborhood. That is, whenever the inequality

$$1 - b_1 < v < 1 + b_2$$

is satisfied, another inequality of the form

$$3 - a_1 < 1 + v < 3 + a_2$$

or $$2 - a_1 < v < 2 + a_2$$

must also be satisfied. The *only* way to achieve this result is to choose $b_1 = a_1 - 1$ and $b_2 = a_2 + 1$. This would imply that the neighborhood of 1 is to be the open interval $(2 - a_1, 2 + a_2)$. According to the definition of a limit, however, a_1 and a_2 can be made arbitrarily small, say, $a_1 = a_2 = 0.1$. In that case, the last-mentioned interval will turn out to be (1.9, 2.1) which lies entirely to the right of the point $v = 1$ on the horizontal axis and, hence, does not even qualify as a neighborhood of 1. Thus the definition of a limit cannot be satisfied by the number 3. A similar procedure can be employed to show that *any* number other than 2 will contradict the definition of a limit in the present case.

In general, if one number satisfies the definition of a limit of q as $v \to N$, then no other number can. If a limit exists, it will be unique.

EXERCISE 6.4

1 Given the function $q = (v^2 + v - 56)/(v - 7)$, $(v \neq 7)$, find the left-side limit and the right-side limit of q as v approaches 7. Can we conclude from these answers that q has a limit as v approaches 7?

2 Given $q = [(v + 2)^3 - 8]/v$, $(v \neq 0)$, find:
(*a*) $\lim_{v \to 0} q$ (*b*) $\lim_{v \to 2} q$ (*c*) $\lim_{v \to a} q$

3 Given $q = 5 - 1/v$, $(v \neq 0)$, find:
(*a*) $\lim_{v \to +\infty} q$ (*b*) $\lim_{v \to -\infty} q$

4 Use Fig. 6.3 to show that we *cannot* consider the number $(L + a_2)$ as the limit of q as v tends to N.

6.5 DIGRESSION ON INEQUALITIES AND ABSOLUTE VALUES

We have encountered inequality signs many times before. In the discussion of the last section, we also applied mathematical operations to inequalities. In transforming (6.7′) into (6.7″), for example, we subtracted 1 from each side of the inequality. What rules of operations apply to inequalities (as opposed to equations)?

Rules of Inequalities

To begin with, let us state an important property of inequalities: inequalities are *transitive*. This means that, if $a > b$ and if $b > c$, then $a > c$. Since equalities (equations) are also transitive, the transitivity property should apply to "weak" inequalities ($\geq$ or $\leq$) as well as to "strict" ones ($>$ or $<$). Thus we have

$$\begin{aligned} a > b,\, b > c \quad &\Rightarrow \quad a > c \\ a \geq b,\, b \geq c \quad &\Rightarrow \quad a \geq c \end{aligned}$$

This property is what makes possible the writing of a *continued inequality*, such as $3 < a < b < 8$ or $7 \leq x \leq 24$. (In writing a continued inequality, the inequality signs are as a rule arranged in the same direction, usually with the smallest number on the left.)

The most important rules of inequalities are those governing the addition (subtraction) of a number to (from) an inequality, the multiplication or division of an inequality by a number, and the squaring of an inequality. Specifically, these rules are as follows.

Rule I (addition and subtraction) $\quad a > b \quad \Rightarrow \quad a \pm k > b \pm k$

An inequality will continue to hold if an equal quantity is added to or subtracted from each side. This rule may be generalized thus: If $a > b > c$, then $a \pm k > b \pm k > c \pm k$.

Rule II (multiplication and division)

$$a > b \quad \Rightarrow \quad \begin{cases} ka > kb & (k > 0) \\ ka < kb & (k < 0) \end{cases}$$

The multiplication of both sides by a *positive* number preserves the inequality, but a *negative* multiplier will cause the *sense* (or *direction*) of the inequality to be reversed.

Example 1 Since $6 > 5$, multiplication by 3 will yield $3(6) > 3(5)$, or $18 > 15$; but multiplication by -3 will result in $(-3)6 < (-3)5$, or $-18 < -15$.

Division of an inequality by a number n is equivalent to multiplication by the number $1/n$; therefore the rule on division is subsumed under the rule on multiplication.

Rule III **(squaring)** $a > b, (b \geq 0) \quad \Rightarrow \quad a^2 > b^2$
If its two sides are both nonnegative, the inequality will continue to hold when both sides are squared.

Example 2 Since $4 > 3$ and since both sides are positive, we have $4^2 > 3^2$, or $16 > 9$. Similarly, since $2 > 0$, it follows that $2^2 > 0^2$, or $4 > 0$.

The above three rules have been stated in terms of strict inequalities, but their validity is unaffected if the $>$ signs are replaced by $\geq$ signs.

Absolute Values and Inequalities

When the domain of a variable x is an open interval (a, b), the domain may be denoted by the set $\{x \mid a < x < b\}$ or, more simply, by the inequality $a < x < b$. Similarly, if it is a closed interval $[a, b]$, it may be expressed by the weak inequality $a \leq x \leq b$. In the special case of an interval of the form $(-a, a)$—say, $(-10, 10)$—it may be represented either by the inequality $-10 < x < 10$ or, alternatively, by the inequality

$$|x| < 10$$

where the symbol $|x|$ denotes the *absolute value* (or *numerical value*) of x.

For any real number n, the absolute value of n is defined as follows:*

$$(6.8) \qquad |n| \equiv \begin{cases} n & (\text{if } n > 0) \\ -n & (\text{if } n < 0) \\ 0 & (\text{if } n = 0) \end{cases}$$

Note that, if $n = 15$, then $|15| = 15$; but if $n = -15$, we find

$$|-15| = -(-15) = 15$$

also. In effect, therefore, the absolute value of any real number is simply its numerical value after the sign is removed. For this reason, we always have $|n| = |-n|$. The absolute value of n is also called the *modulus of n*.

Given the expression $|x| = 10$, we may conclude from (6.8) that x must be either 10 or -10. By the same token, the expression $|x| < 10$ means that (1) if $x > 0$, then $x \equiv |x| < 10$, so that x must be less than 10; but also (2) if $x < 0$, then according to (6.8) we have $-x \equiv |x| < 10$, or $x > -10$, so that x must be greater than -10. Hence, by combining the two parts of this result, we see that x must lie within the open interval $(-10, 10)$. In general, we can write

$$(6.9) \qquad |x| < n \quad \Leftrightarrow \quad -n < x < n \qquad (n > 0)$$

* The absolute-value notation is similar to that of a first-order determinant, but these two concepts are entirely different. The definition of a first-order determinant is $|a_{ij}| \equiv a_{ij}$, regardless of the sign of a_{ij}. In the definition of the absolute value $|n|$, the sign of n will make a difference. The context of the discussion would normally make it clear whether an absolute value or a first-order determinant is under consideration.

which can also be extended to weak inequalities as follows:

(6.10) $\quad |x| \le n \quad \Leftrightarrow \quad -n \le x \le n \quad (n \ge 0)$

Because they are themselves numbers, the absolute values of two numbers m and n can be added, subtracted, multiplied, and divided. The following properties characterize absolute values:

$$|m| + |n| \ge |m + n|$$

$$|m| \cdot |n| = |m \cdot n|$$

$$\frac{|m|}{|n|} = \left|\frac{m}{n}\right|$$

The first of these, interestingly, involves an inequality rather than an equation. The reason for this is easily seen: whereas the left-hand expression $|m| + |n|$ is definitely a *sum* of two numerical values (both taken as positive), the expression $|m + n|$ is the numerical value of *either* a sum (if m and n are, say, both positive) *or* a difference (if m and n have opposite signs). Thus the left side may exceed the right side.

Example 3 If $m = 5$ and $n = 3$, then $|m| + |n| = |m + n| = 8$. But if $m = 5$ and $n = -3$, then $|m| + |n| = 5 + 3 = 8$, whereas

$$|m + n| = |5 - 3| = 2$$

is a smaller number.

In the other two properties, on the other hand, it makes no difference whether m and n have identical or opposite signs, since, in taking the absolute value of the product or quotient on the right-hand side, the sign of the latter term will be removed in any case.

Example 4 If $m = 7$ and $n = 8$, then $|m| \cdot |n| = |m \cdot n| = 7(8) = 56$. But even if $m = -7$ and $n = 8$ (opposite signs), we still get the same result from

$$|m| \cdot |n| = |-7| \cdot |8| = 7(8) = 56$$

and $\quad |m \cdot n| = |-7(8)| = 7(8) = 56$

Solution of an Inequality

Like an equation, an inequality containing a variable (say, x) may have a solution; the solution, if it exists, is a set of values of x which make the inequality a true statement. Such a solution will itself usually be in the form of an inequality.

Example 5 Find the solution of the inequality

$$3x - 3 > x + 1$$

As in solving an equation, the variable terms should first be collected on one side of the inequality. By adding $(3 - x)$ to both sides, we obtain

$$3x - 3 + 3 - x > x + 1 + 3 - x$$
$$\text{or} \qquad 2x > 4$$

Multiplying both sides by $\frac{1}{2}$ (which does not reverse the sense of the inequality, because $\frac{1}{2} > 0$) will then yield the solution

$$x > 2$$

which is itself an inequality. This solution is not a single number, but a set of numbers. Therefore we may also express the solution as the set $\{x \mid x > 2\}$ or as the open interval $(2, \infty)$.

Example 6 Solve the inequality $|1 - x| \leq 3$. First, let us get rid of the absolute-value notation by utilizing (6.10). The given inequality is equivalent to the statement that

$$-3 \leq 1 - x \leq 3$$

or, after subtracting 1 from each side,

$$-4 \leq -x \leq 2$$

Multiplying each side by (-1), we then get

$$4 \geq x \geq -2$$

where the sense of inequality has been duly reversed. Writing the smaller number first, we may express the solution in the form of the inequality

$$-2 \leq x \leq 4$$

or in the form of the set $\{x \mid -2 \leq x \leq 4\}$ or the closed interval $[-2, 4]$.

Sometimes, a problem may call for the satisfaction of several inequalities in several variables simultaneously; then we must solve a system of simultaneous inequalities. This problem arises, for example, in mathematical programming, which will be discussed in the final part of the book.

EXERCISE 6.5

1 Solve the following inequalities:

(*a*) $3x - 1 < 7x + 2$ (*c*) $5x + 1 < x + 3$

(*b*) $2x + 5 < x - 4$ (*d*) $2x - 1 < 6x + 5$

2 If $7x - 3 < 0$ and $7x > 0$, express these in a continued inequality and find its solution.

3 Solve the following:

(*a*) $|x + 1| < 6$ (*b*) $|4 - 3x| < 2$ (*c*) $|2x + 3| \leq 5$

6.6 LIMIT THEOREMS

Our interest in rates of change led us to the consideration of the concept of derivative, which, being in the nature of the limit of a difference quotient, in turn prompted us to study questions of the existence and evaluation of a limit. The basic process of limit evaluation, as illustrated in Sec. 6.4, involves letting the variable v approach a particular number (say, N) and observing the value which q approaches. When actually evaluating the limit of a function, however, we may draw upon certain established limit theorems, which can materially simplify the task, especially for complicated functions.

Theorems Involving a Single Function

When a single function $q = g(v)$ is involved, the following theorems are applicable.

Theorem I If $q = av + b$, then $\lim_{v \to N} q = aN + b$ (a and b are constants).

Example 1 Given $q = 5v + 7$, we have $\lim_{v \to 2} q = 5(2) + 7 = 17$. Similarly, $\lim_{v \to 0} q = 5(0) + 7 = 7$.

Theorem II If $q = g(v) = b$, then $\lim_{v \to N} q = b$.

This theorem, which says that the limit of a constant function is the constant in that function, is merely a special case of Theorem I, with $a = 0$. (You have already encountered an example of this case in Exercise 6.2-3.)

Theorem III If $q = v$, then $\lim_{v \to N} q = N$.

If $q = v^k$, then $\lim_{v \to N} q = N^k$.

Example 2 Given $q = v^3$, we have $\lim_{v \to 2} q = (2)^3 = 8$.

You may have noted that, in the above three theorems, what is done to find the limit of q as $v \to N$ is indeed to let $v = N$. But these are special cases, and they do not vitiate the general rule that "$v \to N$" does not mean "$v = N$."

Theorems Involving Two Functions

If we have two functions of the same independent variable v, $q_1 = g(v)$ and $q_2 = h(v)$, and if *both* functions possess limits as follows:

$$\lim_{v \to N} q_1 = L_1 \qquad \lim_{v \to N} q_2 = L_2$$

where L_1 and L_2 are two *finite* numbers, the following theorems are applicable.

Theorem IV (sum-difference limit theorem)

$$\lim_{v \to N} (q_1 \pm q_2) = L_1 \pm L_2$$

The limit of a sum (difference) of two functions is the sum (difference) of their respective limits.

In particular, we note that

$$\lim_{v \to N} 2q_1 = \lim_{v \to N} (q_1 + q_1) = L_1 + L_1 = 2L_1$$

which is in line with Theorem I.

Theorem V (product limit theorem)

$$\lim_{v \to N} (q_1 q_2) = L_1 L_2$$

The limit of a product of two functions is the product of their limits.

Applied to the square of a function, this gives

$$\lim_{v \to N} (q_1 q_1) = L_1 L_1 = L_1^2$$

which is in line with Theorem III.

Theorem VI (quotient limit theorem)

$$\lim_{v \to N} \frac{q_1}{q_2} = \frac{L_1}{L_2} \qquad (L_2 \neq 0)$$

The limit of a quotient of two functions is the quotient of their limits. Naturally, the limit L_2 is restricted to be nonzero; otherwise the quotient is undefined.

Example 3 Find $\lim_{v \to 0} (1 + v)/(2 + v)$. Since we have here $\lim_{v \to 0} (1 + v) = 1$ and $\lim_{v \to 0} (2 + v) = 2$, the desired limit is $\frac{1}{2}$.

Remember that L_1 and L_2 represent finite numbers; otherwise these theorems do not apply. In the case of Theorem VI, furthermore, L_2 must be nonzero as well. If these restrictions are not satisfied, we must fall back on the method of limit evaluation illustrated in Examples 2 and 3 in Sec. 6.4, which relate to the cases, respectively, of L_2 being zero and of L_2 being infinite.

Limit of a Polynomial Function

With the above limit theorems at our disposal, we can easily evaluate the limit of any polynomial function

$$(6.11) \qquad q = g(v) = a_0 + a_1 v + a_2 v^2 + \cdots + a_n v^n$$

as v tends to the number N. Since the limits of the separate terms are, respectively,

$$\lim_{v \to N} a_0 = a_0 \qquad \lim_{v \to N} a_1 v = a_1 N \qquad \lim_{v \to N} a_2 v^2 = a_2 N^2 \qquad \text{(etc.)}$$

the limit of the polynomial function is (by the sum limit theorem)

$$\lim_{v \to N} q = a_0 + a_1 N + a_2 N^2 + \cdots + a_n N^n \tag{6.12}$$

This limit is also, we note, actually equal to $g(N)$, that is, equal to the value of the function in (6.11) when $v = N$. This particular result will prove important in discussing the concept of *continuity* of the polynomial function.

EXERCISE 6.6

1 Find the limits of the function $q = 8 - 9v + v^2$:
(*a*) As $v \to 0$ (*b*) As $v \to 3$ (*c*) As $v \to -1$

2 Find the limits of $q = (v + 2)(v - 3)$:
(*a*) As $v \to -1$ (*b*) As $v \to 0$ (*c*) As $v \to 4$

3 Find the limits of $q = (3v + 5)/(v + 2)$:
(*a*) As $v \to 0$ (*b*) As $v \to 5$ (*c*) As $v \to -1$

6.7 CONTINUITY AND DIFFERENTIABILITY OF A FUNCTION

The preceding discussion of the concept of limit and its evaluation can now be used to define the continuity and differentiability of a function. These notions bear directly on the derivative of the function, which is what interests us.

Continuity of a Function

When a function $q = g(v)$ possesses a limit as v tends to the point N in the domain, and when this limit is also equal to $g(N)$—that is, equal to the value of the function at $v = N$—the function is said to be *continuous* at N. As stated above, the term *continuity* involves no less than three requirements: (1) the point N must be in the domain of the function; i.e., $g(N)$ is defined; (2) the function must have a limit as $v \to N$; i.e., $\lim_{v \to N} g(v)$ exists; and (3) that limit must be equal in value to $g(N)$; i.e., $\lim_{v \to N} g(v) = g(N)$.

It is important to note that while—in discussing the limit of the curve in Fig. 6.3—the point (N, L) was excluded from consideration, we are no longer excluding it in the present context. Rather, as the third requirement specifically states, the point (N, L) must be on the graph of the function before the function can be considered as continuous at point N.

Let us check whether the functions shown in Fig. 6.2 are continuous. In diagram a, all three requirements are met at point N. Point N is in the domain; q

has the limit L as $v \to N$; and the limit L happens also to be the value of the function at N. Thus, the function represented by that curve is continuous at N. The same is true of the function depicted in Fig. 6.2*b*, since L is the limit of the function as v approaches the value N in the domain, and since L is also the value of the function at N. This last graphic example should suffice to establish that the continuity of a function at point N does *not* necessarily imply that the graph of the function is "smooth" at $v = N$, for the point (N, L) in Fig. 6.2*b* is actually a "sharp" point and yet the function is continuous at that value of v.

When a function $q = g(v)$ is continuous at all values of v in the interval (a, b), it is said to be continuous in that interval. If the function is continuous at all points in a subset S of the domain (where the subset S may be the union of several disjoint intervals), it is said to be continuous in S. And, finally, if the function is continuous at all points in its domain, we say that it is continuous in its domain. Even in this latter case, however, the graph of the function may nevertheless show a discontinuity (a gap) at some value of v, say, at $v = 5$, if that value of v is *not* in its domain.

Again referring to Fig. 6.2, we see that in diagram *c* the function is *discontinuous* at N because a limit does not exist at that point, in violation of the second requirement of continuity. Nevertheless, the function does satisfy the requirements of continuity in the interval $(0, N)$ of the domain, as well as in the interval $[N, \infty)$. Diagram *d* obviously is also discontinuous at $v = N$. This time, discontinuity emanates from the fact that N is excluded from the domain, in violation of the first requirement of continuity.

On the basis of the graphs in Fig. 6.2, it appears that sharp points are consistent with continuity, as in diagram *b*, but that gaps are taboo, as in diagrams *c* and *d*. This is indeed the case. Roughly speaking, therefore, a function that is continuous in a particular interval is one whose graph can be drawn for the said interval without lifting the pencil or pen from the paper—a feat which is possible even if there are sharp points, but impossible when gaps occur.

Polynomial and Rational Functions

Let us now consider the continuity of certain frequently encountered functions. For any polynomial function, such as $q = g(v)$ in (6.11), we have found from (6.12) that $\lim_{v \to N} q$ exists and is equal to the value of the function at N. Since N is a point (any point) in the domain of the function, we can conclude that any polynomial function is continuous in its domain. This is a very useful piece of information, because polynomial functions will be encountered very often.

What about rational functions? Regarding continuity, there exists an interesting theorem (the continuity theorem) which states that the sum, difference, product, and quotient of any finite number of functions that are continuous in the domain are, respectively, also continuous in the domain. As a result, any rational function (a quotient of two polynomial functions) must also be continuous in its domain.

Example 1 The rational function

$$q = g(v) = \frac{4v^2}{v^2 + 1}$$

is defined for all finite real numbers; thus its domain consists of the interval $(-\infty, \infty)$. For any number N in the domain, the limit of q is (by the quotient limit theorem)

$$\lim_{v \to N} q = \frac{\lim_{v \to N} (4v^2)}{\lim_{v \to N} (v^2 + 1)} = \frac{4N^2}{N^2 + 1}$$

which is equal to $g(N)$. Thus the three requirements of continuity are all met at N. Moreover, we note that N can represent any point in the domain of this function; consequently, this function is continuous in its domain.

Example 2 The rational function

$$q = \frac{v^3 + v^2 - 4v - 4}{v^2 - 4}$$

is not defined at $v = 2$ and at $v = -2$. Since those two values of v are not in the domain, the function is discontinuous at $v = -2$ and $v = 2$, despite the fact that a limit of q exists as $v \to -2$ or 2. Graphically, this function will display a gap at each of these two values of v. But for other values of v (those which *are* in the domain), this function is continuous.

Differentiability of a Function

The previous discussion has provided us with the tools for ascertaining whether any function has a limit as its independent variable approaches some specific value. Thus we can try to take the limit of any function $y = f(x)$ as x approaches some chosen value, say, x_0. However, we can also apply the "limit" concept at a different level and take the limit of the difference quotient of that function, $\Delta y/\Delta x$, as Δx approaches zero. The outcomes of limit-taking at these two different levels relate to two different, though related, properties of the function f.

Taking the limit of the function $y = f(x)$ itself, we can, in line with the discussion of the preceding subsection, examine whether the function f is *continuous* at $x = x_0$. The conditions for continuity are (1) $x = x_0$ must be in the domain of the function f, (2) y must have a limit as $x \to x_0$, and (3) the said limit must be equal to $f(x_0)$. When these are satisfied, we can write

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \text{[continuity condition]} \tag{6.13}$$

When the "limit" concept is applied to the difference quotient $\Delta y/\Delta x$ as $\Delta x \to 0$, on the other hand, we deal instead with the question of whether the function f is *differentiable* at $x = x_0$, i.e., whether the derivative dy/dx exists at

$x = x_0$, or whether $f'(x_0)$ exists. The term "differentiable" is used here because the process of obtaining the derivative dy/dx is known as *differentiation* (also called *derivation*). Since $f'(x_0)$ exists if and only if the limit of $\Delta y/\Delta x$ exists at $x = x_0$ as $\Delta x \to 0$, the symbolic expression of the differentiability of f is

$$(6.14) \qquad f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\equiv \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad \text{[differentiability condition]}$$

These two properties, continuity and differentiability, are very intimately related to each other—the continuity of f is a *necessary* condition for its differentiability (although, as we shall see later, this condition is *not sufficient*). What this means is that, to be differentiable at $x = x_0$, the function must first pass the test of being continuous at $x = x_0$. To prove this, we shall demonstrate that, given a function $y = f(x)$, its continuity at $x = x_0$ follows from its differentiability at $x = x_0$, i.e., condition (6.13) follows from condition (6.14). Before doing this, however, let us simplify the notation somewhat by (1) replacing x_0 with the symbol N and (2) replacing $(x_0 + \Delta x)$ with the symbol x. The latter is justifiable because the postchange value of x can be any number (depending on the magnitude of the change) and hence is a variable denotable by x. The equivalence of the two notation systems is shown in Fig. 6.4, where the old notations appear (in brackets) alongside the new. Note that, with the notational change, Δx now becomes $(x - N)$, so that the expression "$\Delta x \to 0$" becomes

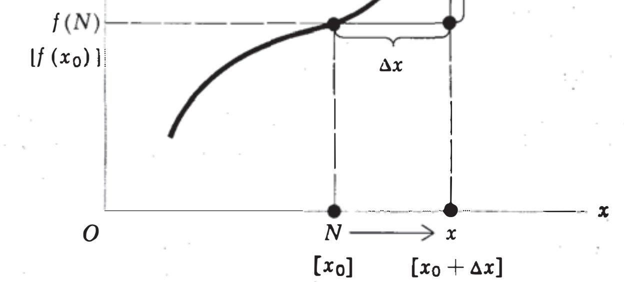

Figure 6.4

"$x \to N$," which is analogous to the expression $v \to N$ used before in connection with the function $q = g(v)$. Accordingly, (6.13) and (6.14) can now be rewritten, respectively, as

$$\lim_{x \to N} f(x) = f(N) \tag{6.13'}$$

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \tag{6.14'}$$

What we want to show is, therefore, that the continuity condition (6.13′) follows from the differentiability condition (6.14′). First, since the notation $x \to N$ implies that $x \neq N$, so that $x - N$ is a nonzero number, it is permissible to write the following identity:

$$f(x) - f(N) \equiv \frac{f(x) - f(N)}{x - N}(x - N) \tag{6.15}$$

Taking the limit of each side of (6.15) as $x \to N$ yields the following results:

$$\text{Left side} = \lim_{x \to N} f(x) - \lim_{x \to N} f(N) \qquad \text{[difference limit theorem]}$$

$$= \lim_{x \to N} f(x) - f(N) \qquad [f(N) \text{ is a constant}]$$

$$\text{Right side} = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} \lim_{x \to N} (x - N) \qquad \text{[product limit theorem]}$$

$$= f'(N)\left(\lim_{x \to N} x - \lim_{x \to N} N\right) \qquad \text{[by (6.14′) and difference limit theorem]}$$

$$= f'(N)(N - N) = 0$$

Note that we could not have written these results, if condition (6.14′) had not been granted, for if $f'(N)$ did not exist, then the right-side expression (and hence also the left-side expression) in (6.15) would not possess a limit. If $f'(N)$ does exist, however, the two sides will have limits as shown above. Moreover, when the left-side result and the right-side result are equated, we get $\lim_{x \to N} f(x) - f(N) = 0$, which is identical with (6.13′). Thus we have proved that continuity, as shown in (6.13′), follows from differentiability, as shown in (6.14′). In general, if a function is differentiable at every point in its domain, we may conclude that it must be continuous in its domain.

Although differentiability implies continuity, the converse is not true. That is, continuity is a *necessary*, but *not* a *sufficient*, condition for differentiability. To demonstrate this, we merely have to produce a counterexample. Let us consider the function

$$y = f(x) = |x - 2| + 1 \tag{6.16}$$

which is graphed in Fig. 6.5. As can be readily shown, this function is not differentiable, though continuous, when $x = 2$. That the function is continuous at $x = 2$ is easy to establish. First, $x = 2$ is in the domain of the function. Second, the limit of y exists as x tends to 2; to be specific, $\lim_{x \to 2^+} y = \lim_{x \to 2^-} y = 1$. Third, $f(2)$ is also found to be 1. Thus all three requirements of continuity are met. To show that the function f is *not* differentiable at $x = 2$, we must show that the limit of the difference quotient

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{|x - 2| + 1 - 1}{x - 2} = \lim_{x \to 2} \frac{|x - 2|}{x - 2}$$

does *not* exist. This involves the demonstration of a disparity between the left-side and the right-side limits. Since, in considering the right-side limit, x must exceed 2, according to the definition of absolute value in (6.8) we have $|x - 2| = x - 2$. Thus the right-side limit is

$$\lim_{x \to 2^+} \frac{|x - 2|}{x - 2} = \lim_{x \to 2^+} \frac{x - 2}{x - 2} = \lim_{x \to 2^+} 1 = 1$$

On the other hand, in considering the left-side limit, x must be less than 2; thus, according to (6.8), $|x - 2| = -(x - 2)$. Consequently, the left-side limit is

$$\lim_{x \to 2^-} \frac{|x - 2|}{x - 2} = \lim_{x \to 2^-} \frac{-(x - 2)}{x - 2} = \lim_{x \to 2^-} (-1) = -1$$

which is different from the right-side limit. This shows that continuity does not guarantee differentiability. In sum, all differentiable functions are continuous, but not all continuous functions are differentiable.

In Fig. 6.5, the nondifferentiability of the function at $x = 2$ is manifest in the fact that the point (2, 1) has no tangent line defined, and hence no definite slope can be assigned to the point. Specifically, to the left of that point, the curve has a

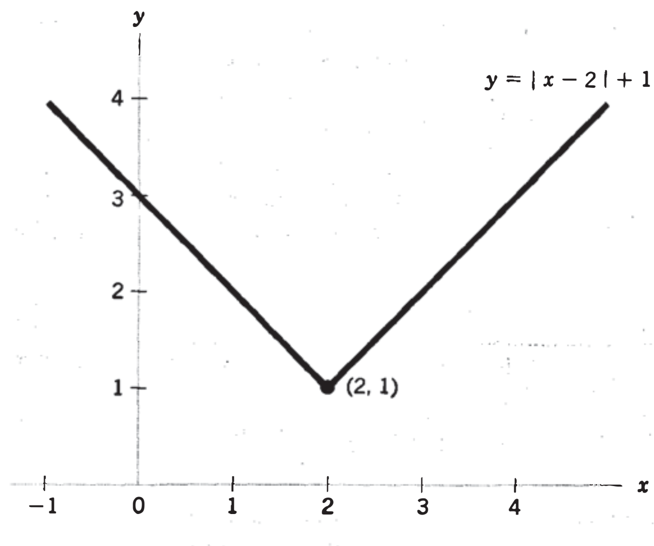

Figure 6.5

slope of -1, but to the right it has a slope of $+1$, and the slopes on the two sides display no tendency to approach a common magnitude at $x = 2$. The point (2, 1) is, of course, a special point; it is the only sharp point on the curve. At other points on the curve, the derivative is defined and the function is differentiable. More specifically, the function in (6.16) can be divided into two linear functions as follows:

$$\text{Left part:} \quad y = -(x-2)+1 = 3-x \qquad (x \le 2)$$

$$\text{Right part:} \quad y = (x-2)+1 = x-1 \qquad (x > 2)$$

The left part is differentiable in the interval $(-\infty, 2)$, and the right part is differentiable in the interval $(2, \infty)$ in the domain.

In general, differentiability is a more restrictive condition than continuity, because it requires something beyond continuity. Continuity at a point only rules out the presence of a gap, whereas differentiability rules out "sharpness" as well. Therefore, differentiability calls for "smoothness" of the function (curve) as well as its continuity. Most of the *specific* functions employed in economics have the property that they are differentiable everywhere. When *general* functions are used, moreover, they are often assumed to be everywhere differentiable, as we shall do in the subsequent discussion.

EXERCISE 6.7

1 A function $y = f(x)$ is discontinuous at $x = x_0$ when *any* of the three requirements for continuity is violated at $x = x_0$. Construct three graphs to illustrate the violation of each of those requirements.

2 Taking the set of all finite real numbers as the domain of the function $q = g(v) = v^2 - 7v - 3$:

(*a*) Find the limit of q as v tends to N (a finite real number).

(*b*) Check whether this limit is equal to $g(N)$.

(*c*) Check whether the function is continuous at N and continuous in its domain.

3 Given the function $q = g(v) = \dfrac{v+2}{v^2+2}$:

(*a*) Use the limit theorems to find $\lim_{v \to N} q$, N being a finite real number.

(*b*) Check whether this limit is equal to $g(N)$.

(*c*) Check the continuity of the function $g(v)$ at N and in its domain $(-\infty, \infty)$.

4 Given $y = f(x) = \dfrac{x^2 + x - 20}{x - 4}$:

(*a*) Is it possible to apply the quotient limit theorem to find the limit of this function as $x \to 4$?

(*b*) Is this function continuous at $x = 4$? Why?

(*c*) Find a function which, for $x \neq 4$, is equivalent to the above function, and obtain from the equivalent function the limit of y as $x \to 4$.

5 In the rational function in Example 2, the numerator is evenly divisible by the denominator, and the quotient is $v + 1$. Can we for that reason replace that function outright by $q = v + 1$? Why or why not?

6 On the basis of the graphs of the six functions in Fig. 2.8, would you conclude that each such function is differentiable at every point in its domain? Explain.

CHAPTER

SEVEN

RULES OF DIFFERENTIATION AND THEIR USE IN COMPARATIVE STATICS

The central problem of comparative-static analysis, that of finding a rate of change, can be identified with the problem of finding the derivative of some function $y = f(x)$, provided only a small change in x is being considered. Even though the derivative dy/dx is defined as the limit of the difference quotient $q = g(v)$ as $v \to 0$, it is by no means necessary to undertake the process of limit-taking each time the derivative of a function is sought, for there exist various rules of differentiation (derivation) that will enable us to obtain the desired derivatives directly. Instead of going into comparative-static models immediately, therefore, let us begin by learning some rules of differentiation.

7.1 RULES OF DIFFERENTIATION FOR A FUNCTION OF ONE VARIABLE

First, let us discuss three rules that apply, respectively, to the following types of function of a single independent variable: $y = k$ (constant function), $y = x^n$, and $y = cx^n$ (power functions). All these have smooth, continuous graphs and are therefore differentiable everywhere.

Constant-Function Rule

The derivative of a constant function $y = f(x) = k$ is identically zero, i.e., is zero for all values of x. Symbolically, this may be expressed variously as

$$\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dk}{dx} = 0 \qquad \text{or} \qquad f'(x) = 0$$

In fact, we may also write these in the form

$$\frac{d}{dx}y = \frac{d}{dx}f(x) = \frac{d}{dx}k = 0$$

where the derivative symbol has been separated into two parts, d/dx on the one hand, and y [or $f(x)$ or k] on the other. The first part, d/dx, may be taken as an *operator symbol*, which instructs us to perform a particular mathematical operation. Just as the operator symbol $\sqrt{}$ instructs us to take a square root, the symbol d/dx represents an instruction to take the derivative of, or to differentiate, (some function) with respect to the variable x. The function to be operated on (to be differentiated) is indicated in the second part; here it is $y = f(x) = k$.

The proof of the rule is as follows. Given $f(x) = k$, we have $f(N) = k$ for any value of N. Thus the value of $f'(N)$—the value of the derivative at $x = N$—as defined in (6.13) will be

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{k - k}{x - N} = \lim_{x \to N} 0 = 0$$

Moreover, since N represents any value of x at all, the result $f'(N) = 0$ can be immediately generalized to $f'(x) = 0$. This proves the rule.

It is important to distinguish clearly between the statement $f'(x) = 0$ and the similar-looking but different statement $f'(x_0) = 0$. By $f'(x) = 0$, we mean that the derivative function f' has a zero value for *all* values of x; in writing $f'(x_0) = 0$, on the other hand, we are merely associating the zero value of the derivative with a particular value of x, namely, $x = x_0$.

As discussed before, the derivative of a function has its geometric counterpart in the slope of the curve. The graph of a constant function, say, a fixed-cost function $C_F = f(Q) = \$1200$, is a horizontal straight line with a zero slope throughout. Correspondingly, the derivative must also be zero for all values of Q:

$$\frac{d}{dQ}C_F = \frac{d}{dQ}1200 = 0 \qquad \text{or} \qquad f'(Q) = 0$$

Power-Function Rule

The derivative of a power function $y = f(x) = x^n$ is nx^{n-1}. Symbolically, this is expressed as

(7.1) $$\frac{d}{dx}x^n = nx^{n-1} \qquad \text{or} \qquad f'(x) = nx^{n-1}$$

Example 1 The derivative of $y = x^3$ is $\dfrac{dy}{dx} = \dfrac{d}{dx}x^3 = 3x^2$.

Example 2 The derivative of $y = x^9$ is $\dfrac{d}{dx}x^9 = 9x^8$.

This rule is valid for any real-valued power of x; that is, the exponent can be any real number. But we shall prove it only for the case where n is some positive integer. In the simplest case, that of $n = 1$, the function is $f(x) = x$, and according to the rule, the derivative is

$$f'(x) = \frac{d}{dx}x = 1(x^0) = 1$$

The proof of this result follows easily from the definition of $f'(N)$ in (6.14'). Given $f(x) = x$, the derivative value at any value of x, say, $x = N$, is

$$f'(N) = \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x - N}{x - N} = \lim_{x \to N} 1 = 1$$

Since N represents any value of x, it is permissible to write $f'(x) = 1$. This proves the rule for the case of $n = 1$. As the graphical counterpart of this result, we see that the function $y = f(x) = x$ plots as a 45° line, and it has a slope of $+1$ throughout.

For the cases of larger integers, $n = 2, 3, \ldots$, let us first note the following identities:

$$\frac{x^2 - N^2}{x - N} = x + N \qquad \text{[2 terms on the right]}$$

$$\frac{x^3 - N^3}{x - N} = x^2 + Nx + N^2 \qquad \text{[3 terms on the right]}$$

$$\vdots$$

(7.2) $$\frac{x^n - N^n}{x - N} = x^{n-1} + Nx^{n-2} + N^2x^{n-3} + \cdots + N^{n-1}$$

[n terms on the right]

On the basis of (7.2), we can express the derivative of a power function $f(x) = x^n$ at $x = N$ as follows:

(7.3) $$\begin{aligned} f'(N) &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{x^n - N^n}{x - N} \\ &= \lim_{x \to N} (x^{n-1} + Nx^{n-2} + \cdots + N^{n-1}) \quad \text{[by (7.2)]} \\ &= \lim_{x \to N} x^{n-1} + \lim_{x \to N} Nx^{n-2} + \cdots + \lim_{x \to N} N^{n-1} \quad \text{[sum limit theorem]} \\ &= N^{n-1} + N^{n-1} + \cdots + N^{n-1} \quad \text{[a total of } n \text{ terms]} \\ &= nN^{n-1} \end{aligned}$$

Again, N is any value of x; thus this last result can be generalized to

$$f'(x) = nx^{n-1}$$

which proves the rule for n, any positive integer.

As mentioned above, this rule applies even when the exponent n in the power expression x^n is not a positive integer. The following examples serve to illustrate its application to the latter cases.

Example 3 Find the derivative of $y = x^0$. Applying (7.1), we find

$$\frac{d}{dx}x^0 = 0(x^{-1}) = 0$$

Example 4 Find the derivative of $y = 1/x^3$. This involves the reciprocal of a power, but by rewriting the function as $y = x^{-3}$, we can again apply (7.1) to get the derivative:

$$\frac{d}{dx}x^{-3} = -3x^{-4} \qquad \left[= \frac{-3}{x^4}\right]$$

Example 5 Find the derivative of $y = \sqrt{x}$. A square root is involved in this case, but since $\sqrt{x} = x^{1/2}$, the derivative can be found as follows:

$$\frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} \qquad \left[= \frac{1}{2\sqrt{x}}\right]$$

Derivatives are themselves functions of the independent variable x. In Example 1, for instance, the derivative is $dy/dx = 3x^2$, or $f'(x) = 3x^2$, so that a different value of x will result in a different value of the derivative, such as

$$f'(1) = 3(1)^2 = 3 \qquad f'(2) = 3(2)^2 = 12$$

These specific values of the derivative can be expressed alternatively as

$$\left.\frac{dy}{dx}\right|_{x=1} = 3 \qquad \left.\frac{dy}{dx}\right|_{x=2} = 12$$

but the notations $f'(1)$ and $f'(2)$ are obviously preferable because of their simplicity.

It is of the utmost importance to realize that, to find the derivative values $f'(1)$, $f'(2)$, etc., we must *first* differentiate the function $f(x)$, in order to get the derivative function $f'(x)$, and *then* let x assume specific values in $f'(x)$. To substitute specific values of x into the primitive function $f(x)$ prior to differentiation is definitely not permissible. As an illustration, if we let $x = 1$ in the function of Example 1 before differentiation, the function will degenerate into $y = x = 1$ —a constant function—which will yield a zero derivative rather than the correct answer of $f'(x) = 3x^2$.

Power-Function Rule Generalized

When a multiplicative constant c appears in the power function, so that $f(x) = cx^n$, its derivative is

$$\frac{d}{dx}cx^n = cnx^{n-1} \qquad \text{or} \qquad f'(x) = cnx^{n-1}$$

This result shows that, in differentiating cx^n, we can simply retain the multiplicative constant c intact and then differentiate the term x^n according to (7.1).

Example 6 Given $y = 2x$, we have $dy/dx = 2x^0 = 2$.

Example 7 Given $f(x) = 4x^3$, the derivative is $f'(x) = 12x^2$.

Example 8 The derivative of $f(x) = 3x^{-2}$ is $f'(x) = -6x^{-3}$.

For a proof of this new rule, consider the fact that for any value of x, say, $x = N$, the value of the derivative of $f(x) = cx^n$ is

$$\begin{aligned} f'(N) &= \lim_{x \to N} \frac{f(x) - f(N)}{x - N} = \lim_{x \to N} \frac{cx^n - cN^n}{x - N} = \lim_{x \to N} c\left(\frac{x^n - N^n}{x - N}\right) \\ &= \lim_{x \to N} c \lim_{x \to N} \frac{x^n - N^n}{x - N} \qquad \text{[product limit theorem]} \\ &= c \lim_{x \to N} \frac{x^n - N^n}{x - N} \qquad \text{[limit of a constant]} \\ &= cnN^{n-1} \qquad \text{[from (7.3)]} \end{aligned}$$

In view that N is any value of x, this last result can be generalized immediately to $f'(x) = cnx^{n-1}$, which proves the rule.

EXERCISE 7.1

1 Find the derivative of each of the following functions:
(a) $y = x^{13}$ (c) $y = 7x^6$ (e) $w = -4u^{1/2}$
(b) $y = 63$ (d) $w = 3u^{-1}$

2 Find the following:
(a) $\frac{d}{dx}(-x^{-4})$ (c) $\frac{d}{dw}9w^4$ (e) $\frac{d}{du}au^b$
(b) $\frac{d}{dx}7x^{1/3}$ (d) $\frac{d}{dx}cx^2$

3 Find $f'(1)$ and $f'(2)$ from the following functions:
(a) $y = f(x) = 18x$ (c) $f(x) = -5x^{-2}$ (e) $f(w) = 6w^{1/3}$
(b) $y = f(x) = cx^3$ (d) $f(x) = \frac{3}{4}x^{4/3}$

4 Graph a function $f(x)$ that gives rise to the derivative function $f'(x) = 0$. Then graph a function $g(x)$ characterized by $f'(x_0) = 0$.

7.2 RULES OF DIFFERENTIATION INVOLVING TWO OR MORE FUNCTIONS OF THE SAME VARIABLE

The three rules presented in the preceding section are each concerned with a single given function $f(x)$. Now suppose that we have two *differentiable* functions of the same variable x, say, $f(x)$ and $g(x)$, and we want to differentiate the sum, difference, product, or quotient formed with these two functions. In such circumstances, are there appropriate rules that apply? More concretely, given two functions—say, $f(x) = 3x^2$ and $g(x) = 9x^{12}$—how do we get the derivative of, say, $3x^2 + 9x^{12}$, or the derivative of $(3x^2)(9x^{12})$?

Sum-Difference Rule

The derivative of a sum (difference) of two functions is the sum (difference) of the derivatives of the two functions:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$$

The proof of this again involves the application of the definition of a derivative and of the various limit theorems. We shall omit the proof and, instead, merely verify its validity and illustrate its application.

Example 1 From the function $y = 14x^3$, we can obtain the derivative $dy/dx = 42x^2$. But $14x^3 = 5x^3 + 9x^3$, so that y may be regarded as the sum of two functions $f(x) = 5x^3$ and $g(x) = 9x^3$. According to the sum rule, we then have

$$\frac{dy}{dx} = \frac{d}{dx}(5x^3 + 9x^3) = \frac{d}{dx}5x^3 + \frac{d}{dx}9x^3 = 15x^2 + 27x^2 = 42x^2$$

which is identical with our earlier result.

This rule, stated above in terms of two functions, can easily be extended to more functions. Thus, it is also valid to write

$$\frac{d}{dx}[f(x) \pm g(x) \pm h(x)] = f'(x) \pm g'(x) \pm h'(x)$$

Example 2 The function cited in Example 1, $y = 14x^3$, can be written as $y = 2x^3 + 13x^3 - x^3$. The derivative of the latter, according to the sum-difference rule, is

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3 + 13x^3 - x^3) = 6x^2 + 39x^2 - 3x^2 = 42x^2$$

which again checks with the previous answer.

This rule is of great practical importance. With it at our disposal, it is now possible to find the derivative of any polynomial function, since the latter is nothing but a sum of power functions.

Example 3 $\frac{d}{dx}(ax^2 + bx + c) = 2ax + b$

Example 4

$$\frac{d}{dx}(7x^4 + 2x^3 - 3x + 37) = 28x^3 + 6x^2 - 3 + 0 = 28x^3 + 6x^2 - 3$$

Note that in the last two examples the constants c and 37 do not really produce any effect on the derivative, because the derivative of a constant term is zero. In contrast to the *multiplicative* constant, which is retained during differentiation, the *additive* constant drops out. This fact provides the mathematical explanation of the well-known economic principle that the fixed cost of a firm does not affect its marginal cost. Given a short-run total-cost function

$$C = Q^3 - 4Q^2 + 10Q + 75$$

the marginal-cost function (for infinitesimal output change) is the limit of the quotient $\Delta C/\Delta Q$, or the derivative of the C function:

$$\frac{dC}{dQ} = 3Q^2 - 8Q + 10$$

whereas the fixed cost is represented by the additive constant 75. Since the latter drops out during the process of deriving dC/dQ, the magnitude of the fixed cost obviously cannot affect the marginal cost.

In general, if a primitive function $y = f(x)$ represents a *total* function, then the derivative function dy/dx is its *marginal* function. Both functions can, of course, be plotted against the variable x graphically; and because of the correspondence between the derivative of a function and the slope of its curve, for each value of x the marginal function should show the slope of the total function at that value of x. In Fig. 7.1a, a linear (constant-slope) total function is seen to have a constant marginal function. On the other hand, the nonlinear (varying-slope) total function in Fig. 7.1b gives rise to a curved marginal function, which lies below (above) the horizontal axis when the total function is negatively (positively) sloped. And, finally, the reader may note from Fig. 7.1c (cf. Fig. 6.5) that "nonsmoothness" of a total function will result in a gap (discontinuity) in the marginal or derivative function. This is in sharp contrast to the everywhere-smooth total function in Fig. 7.1b which gives rise to a continuous marginal function. For this reason, the *smoothness* of a *primitive* function can be linked to the *continuity* of its *derivative* function. In particular, instead of saying that a certain function is smooth (and differentiable) everywhere, we may alternatively characterize it as a function with a continuous derivative function, and refer to it as a *continuously differentiable* function.

Product Rule

The derivative of the product of two (differentiable) functions is equal to the first function times the derivative of the second function plus the second function

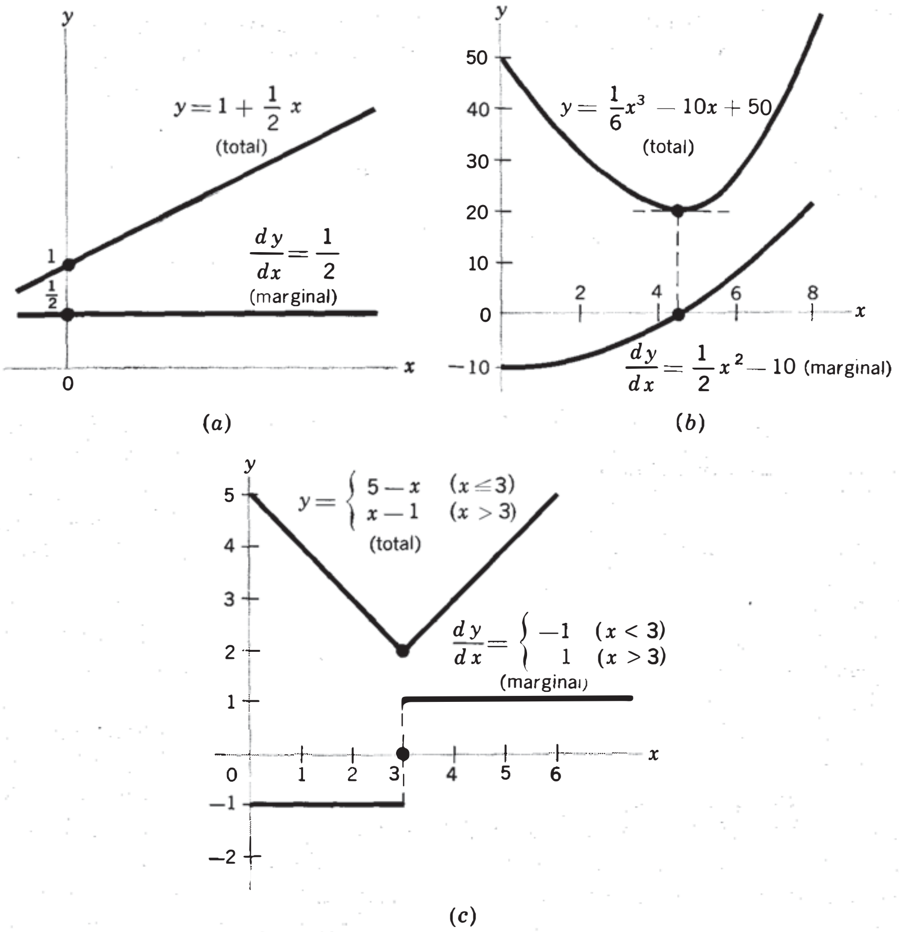

Figure 7.1

times the derivative of the first function:

$$(7.4) \qquad \frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$$
$$= f(x)g'(x) + g(x)f'(x)$$

Example 5 Find the derivative of $y = (2x + 3)(3x^2)$. Let $f(x) = 2x + 3$ and $g(x) = 3x^2$. Then it follows that $f'(x) = 2$ and $g'(x) = 6x$, and according to (7.4) the desired derivative is

$$\frac{d}{dx}[(2x + 3)(3x^2)] = (2x + 3)(6x) + (3x^2)(2) = 18x^2 + 18x$$

This result can be checked by first multiplying out $f(x)g(x)$ and then taking the derivative of the product polynomial. The product polynomial is in this case $f(x)g(x) = (2x + 3)(3x^2) = 6x^3 + 9x^2$, and direct differentiation does yield the same derivative, $18x^2 + 18x$.

The important point to remember is that the derivative of a product of two functions is *not* the simple product of the two separate derivatives. Since this differs from what intuitive generalization leads one to expect, let us produce a proof for (7.4). According to (6.13), the value of the derivative of $f(x)g(x)$ when $x = N$ should be

$$\text{(7.5)} \qquad \frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = \lim_{x \to N} \frac{f(x)g(x) - f(N)g(N)}{x - N}$$

But, by adding *and* subtracting $f(x)g(N)$ in the numerator (thereby leaving the original magnitude unchanged), we can transform the quotient on the right of (7.5) as follows:

$$\frac{f(x)g(x) - f(x)g(N) + f(x)g(N) - f(N)g(N)}{x - N}$$

$$= f(x)\frac{g(x) - g(N)}{x - N} + g(N)\frac{f(x) - f(N)}{x - N}$$

Substituting this for the quotient on the right of (7.5) and taking its limit, we then get

$$\text{(7.5}'\text{)} \qquad \frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = \lim_{x \to N} f(x) \lim_{x \to N} \frac{g(x) - g(N)}{x - N} + \lim_{x \to N} g(N) \lim_{x \to N} \frac{f(x) - f(N)}{x - N}$$

The four limit expressions in (7.5′) are easily evaluated. The first one is $f(N)$, and the third is $g(N)$ (limit of a constant). The remaining two are, according to (6.13), respectively, $g'(N)$ and $f'(N)$. Thus (7.5′) reduces to

$$\text{(7.5}''\text{)} \qquad \frac{d}{dx}[f(x)g(x)]\bigg|_{x=N} = f(N)g'(N) + g(N)f'(N)$$

And, since N represents any value of x, (7.5″) remains valid if we replace every N symbol by x. This proves the rule.

As an extension of the rule to the case of *three* functions, we have

$$\text{(7.6)} \qquad \frac{d}{dx}[f(x)g(x)h(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

In words, the derivative of the product of three functions is equal to the product of the second and third functions times the derivative of the first, plus the product of the first and third functions times the derivative of the second, plus the product of the first and second functions times the derivative of the third. This result can be derived by the repeated application of (7.4). First treat the product $g(x)h(x)$ as a single function, say, $\phi(x)$, so that the original product of three functions will become a product of *two* functions, $f(x)\phi(x)$. To this, (7.4) is applicable. After the derivative of $f(x)\phi(x)$ is obtained, we may reapply (7.4) to the product $g(x)h(x) \equiv \phi(x)$ to get $\phi'(x)$. Then (7.6) will follow. The details are left to you as an exercise.

The validity of a rule is one thing; its serviceability is something else. Why do we need the product rule when we can resort to the alternative procedure of multiplying out the two functions $f(x)$ and $g(x)$ and then taking the derivative of the product directly? One answer to that question is that the alternative procedure is applicable only to *specific* (numerical or parametric) functions, whereas the product rule is applicable even when the functions are given in the *general* form. Let us illustrate with an economic example.

Finding Marginal-Revenue Function from Average-Revenue Function

If we are given an average-revenue (AR) function in specific form,

$$\text{AR} = 15 - Q$$

the marginal-revenue (MR) function can be found by first multiplying AR by Q to get the total-revenue (R) function:

$$R \equiv \text{AR} \cdot Q = (15 - Q)Q = 15Q - Q^2$$

and then differentiating R:

$$\text{MR} \equiv \frac{dR}{dQ} = 15 - 2Q$$

But if the AR function is given in the general form $\text{AR} = f(Q)$, then the total-revenue function will also be in a general form:

$$R \equiv \text{AR} \cdot Q = f(Q) \cdot Q$$

and therefore the "multiply out" approach will be to no avail. However, because R is a product of two functions of Q, namely, $f(Q)$ and Q itself, the product rule may be put to work. Thus we can differentiate R to get the MR function as follows:

$$\text{(7.7)} \qquad \text{MR} \equiv \frac{dR}{dQ} = f(Q) \cdot 1 + Q \cdot f'(Q) = f(Q) + Qf'(Q)$$

However, can such a general result tell us anything significant about the MR? Indeed it can. Recalling that $f(Q)$ denotes the AR function, let us rearrange (7.7) and write

$$\text{(7.7')} \qquad \text{MR} - \text{AR} = \text{MR} - f(Q) = Qf'(Q)$$

This gives us an important relationship between MR and AR: namely, they will always differ by the amount $Qf'(Q)$.

It remains to examine the expression $Qf'(Q)$. Its first component Q denotes output and is always nonnegative. The other component, $f'(Q)$, represents the slope of the AR curve plotted against Q. Since "average revenue" and "price" are but different names for the same thing:

$$\text{AR} \equiv \frac{R}{Q} \equiv \frac{PQ}{Q} \equiv P$$

the AR curve can also be regarded as a curve relating price P to output Q: $P = f(Q)$. Viewed in this light, the AR curve is simply the *inverse* of the demand curve for the product of the firm, i.e., the demand curve plotted after the P and Q axes are reversed. Under pure competition, the AR curve is a horizontal straight line, so that $f'(Q) = 0$ and, from (7.7′), MR − AR = 0 for all possible values of Q. Thus the MR curve and the AR curve must coincide. Under imperfect competition, on the other hand, the AR curve is normally downward-sloping, as in Fig. 7.2, so that $f'(Q) < 0$ and, from (7.7′), MR − AR < 0 for all positive levels of output. In this case, the MR curve must lie below the AR curve.

The conclusion just stated is *qualitative* in nature; it concerns only the relative positions of the two curves. But (7.7′) also furnishes the *quantitative* information that the MR curve will fall short of the AR curve at any output level Q by precisely the amount $Qf'(Q)$. Let us look at Fig. 7.2 again and consider the particular output level N. For that output, the expression $Qf'(Q)$ specifically becomes $Nf'(N)$; if we can find the magnitude of $Nf'(N)$ in the diagram, we shall know how far below the average-revenue point G the corresponding marginal-revenue point must lie.

The magnitude of N is already specified. And $f'(N)$ is simply the slope of the AR curve at point G (where $Q = N$), that is, the slope of the tangent line JM measured by the ratio of two distances OJ/OM. However, we see that $OJ/OM =$

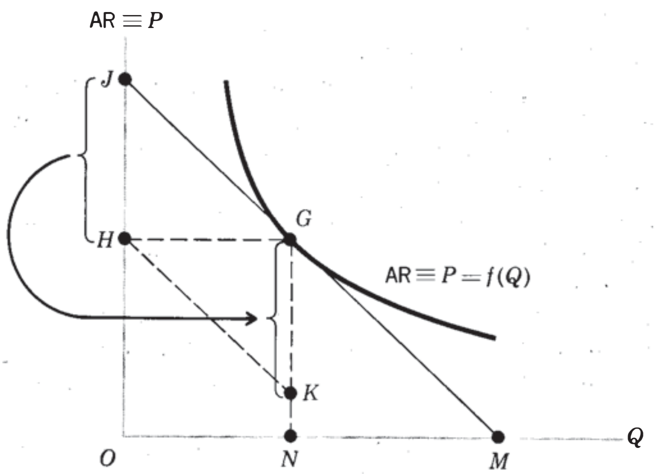

Figure 7.2

HJ/HG; besides, distance HG is precisely the amount of output under consideration, N. Thus the distance $Nf'(N)$, by which the MR curve must lie below the AR curve at output N, is

$$Nf'(N) = HG\frac{HJ}{HG} = HJ$$

Accordingly, if we mark a vertical distance $KG = HJ$ directly below point G, then point K must be a point on the MR curve. (A simple way of accurately plotting KG is to draw a straight line passing through point H and parallel to JG; point K is where that line intersects the vertical line NG.)

The same procedure can be used to locate other points on the MR curve. All we must do, for any chosen point G' on the curve, is first to draw a tangent to the AR curve at G' that will meet the vertical axis at some point J'. Then draw a horizontal line from G' to the vertical axis, and label the intersection with the axis as H'. If we mark a vertical distance $K'G' = H'J'$ directly below point G', then the point K' will be a point on the MR curve. This is the graphical way of deriving an MR curve from a given AR curve. Strictly speaking, the accurate drawing of a tangent line requires a knowledge of the value of the derivative at the relevant output, that is, $f'(N)$; hence the graphical method just outlined cannot quite exist by itself. An important exception is the case of a linear AR curve, where the tangent to any point on the curve is simply the given line itself, so that there is in effect no need to draw any tangent at all. Then the above graphical method will apply in a straightforward way.

Quotient Rule

The derivative of the quotient of two functions, $f(x)/g(x)$, is

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

In the numerator of the right-hand expression, we find two product terms, each involving the derivative of only one of the two original functions. Note that $f'(x)$ appears in the positive term, and $g'(x)$ in the negative term. The denominator consists of the square of the function $g(x)$; that is, $g^2(x) \equiv [g(x)]^2$.

Example 6 $$\frac{d}{dx}\left(\frac{2x-3}{x+1}\right) = \frac{2(x+1)-(2x-3)(1)}{(x+1)^2} = \frac{5}{(x+1)^2}$$

Example 7 $$\frac{d}{dx}\left(\frac{5x}{x^2+1}\right) = \frac{5(x^2+1)-5x(2x)}{(x^2+1)^2} = \frac{5(1-x^2)}{(x^2+1)^2}$$

Example 8 $$\frac{d}{dx}\left(\frac{ax^2+b}{cx}\right) = \frac{2ax(cx)-(ax^2+b)(c)}{(cx)^2} = \frac{c(ax^2-b)}{(cx)^2} = \frac{ax^2-b}{cx^2}$$

This rule can be proved as follows. For any value of $x = N$, we have

$$\text{(7.8)} \qquad \left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=N} = \lim_{x \to N} \frac{f(x)/g(x) - f(N)/g(N)}{x - N}$$

The quotient expression following the limit sign can be rewritten in the form

$$\frac{f(x)g(N) - f(N)g(x)}{g(x)g(N)} \, \frac{1}{x - N}$$

By adding *and* subtracting $f(N)g(N)$ in the numerator and rearranging, we can further transform the expression to

$$\frac{1}{g(x)g(N)}\left[\frac{f(x)g(N) - f(N)g(N) + f(N)g(N) - f(N)g(x)}{x - N}\right]$$

$$= \frac{1}{g(x)g(N)}\left[g(N)\frac{f(x) - f(N)}{x - N} - f(N)\frac{g(x) - g(N)}{x - N}\right]$$

Substituting this result into (7.8) and taking the limit, we then have

$$\left.\frac{d}{dx}\frac{f(x)}{g(x)}\right|_{x=N} = \lim_{x \to N}\frac{1}{g(x)g(N)}\left[\lim_{x \to N} g(N) \lim_{x \to N}\frac{f(x) - f(N)}{x - N}\right.$$

$$\left. - \lim_{x \to N} f(N) \lim_{x \to N}\frac{g(x) - g(N)}{x - N}\right]$$

$$= \frac{1}{g^2(N)}[g(N)f'(N) - f(N)g'(N)] \qquad \text{[by (6.13)]}$$

which can be generalized by replacing the symbol N with x, because N represents any value of x. This proves the quotient rule.

Relationship Between Marginal-Cost and Average-Cost Functions

As an economic application of the quotient rule, let us consider the rate of change of average cost when output varies.

Given a total-cost function $C = C(Q)$, the average-cost (AC) function will be a quotient of two functions of Q, since $AC \equiv C(Q)/Q$, defined as long as $Q > 0$. Therefore, the rate of change of AC with respect to Q can be found by differentiating AC:

$$\text{(7.9)} \qquad \frac{d}{dQ}\frac{C(Q)}{Q} = \frac{[C'(Q)\cdot Q - C(Q)\cdot 1]}{Q^2} = \frac{1}{Q}\left[C'(Q) - \frac{C(Q)}{Q}\right]$$

From this it follows that, for $Q > 0$,

$$\text{(7.10)} \qquad \frac{d}{dQ}\frac{C(Q)}{Q} \gtreqless 0 \quad \text{iff} \quad C'(Q) \gtreqless \frac{C(Q)}{Q}$$

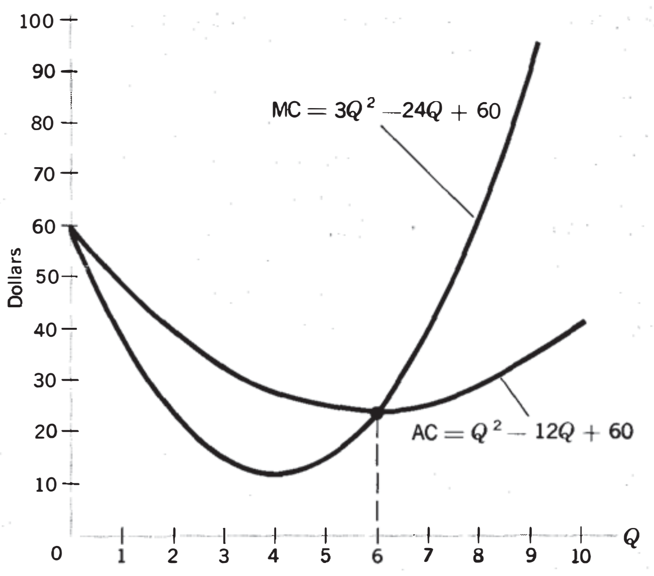

Figure 7.3

Since the derivative $C'(Q)$ represents the marginal-cost (MC) function, and $C(Q)/Q$ represents the AC function, the economic meaning of (7.10) is: The slope of the AC curve will be positive, zero, or negative if and only if the marginal-cost curve lies above, intersects, or lies below the AC curve. This is illustrated in Fig. 7.3, where the MC and AC functions plotted are based on the specific total-cost function

$$C = Q^3 - 12Q^2 + 60Q$$

To the left of $Q = 6$, AC is declining, and thus MC lies below it; to the right, the opposite is true. At $Q = 6$, AC has a slope of zero, and MC and AC have the same value.*

The qualitative conclusion in (7.10) is stated explicitly in terms of cost functions. However, its validity remains unaffected if we interpret $C(Q)$ as *any other* differentiable total function, with $C(Q)/Q$ and $C'(Q)$ as its corresponding average and marginal functions. Thus this result gives us a *general* marginal-average relationship. In particular, we may point out, the fact that MR lies below AR when AR is downward-sloping, as discussed in connection with Fig. 7.2, is nothing but a special case of the general result in (7.10).

* Note that (7.10) does *not* state that, when AC is negatively sloped, MC must also be negatively sloped; it merely says that AC must exceed MC in that circumstance. At $Q = 5$ in Fig. 7.3, for instance, AC is declining but MC is rising, so that their slopes will have opposite signs.

EXERCISE 7.2

1 Given the total-cost function $C = Q^3 - 5Q^2 + 14Q + 75$, write out a variable-cost (VC) function. Find the derivative of the VC function, and interpret the economic meaning of that derivative.

2 Given the average-cost function $\text{AC} = Q^2 - 4Q + 214$, find the MC function. Is the given function more appropriate as a long-run or a short-run function? Why?

3 Differentiate the following by using the product rule:

(*a*) $(9x^2 - 2)(3x + 1)$ (*d*) $(ax - b)(cx^2)$
(*b*) $(3x + 11)(6x^2 - 5x)$ (*e*) $(2 - 3x)(1 + x)(x + 2)$
(*c*) $x^2(4x + 6)$ (*f*) $(x^2 + 3)x^{-1}$

4 (*a*) Given $\text{AR} = 60 - 3Q$, plot the average-revenue curve, and then find the MR curve by the method used in Fig. 7.2.

(*b*) Find the total-revenue function and the marginal-revenue function mathematically from the given AR function.

(*c*) Does the graphically derived MR curve in (*a*) check with the mathematically derived MR function in (*b*)?

(*d*) Comparing the AR and MR functions, what can you conclude about their relative slopes?

5 Provide a mathematical proof for the general result that, given a *linear* average curve, the corresponding marginal curve must have the same vertical intercept but will be twice as steep as the average curve.

6 Prove the result in (7.6) by first treating $g(x)h(x)$ as a single function, $g(x)h(x) \equiv \phi(x)$, and then applying the product rule (7.4).

7 Find the derivatives of:

(*a*) $(x^2 + 3)/x$ (*c*) $4x/(x + 5)$
(*b*) $(x + 7)/x$ (*d*) $(ax^2 + b)/(cx + d)$

8 Given the function $f(x) = ax + b$, find the derivatives of:

(a) $f(x)$ (*b*) $xf(x)$ (*c*) $1/f(x)$ (*d*) $f(x)/x$

7.3 RULES OF DIFFERENTIATION INVOLVING FUNCTIONS OF DIFFERENT VARIABLES

In the preceding section, we discussed the rules of differentiation of a sum, difference, product, or quotient of two (or more) differentiable functions of the same variable. Now we shall consider cases where there are two or more differentiable functions, each of which has a *distinct* independent variable.

Chain Rule

If we have a function $z = f(y)$, where y is in turn a function of another variable x, say, $y = g(x)$, then the derivative of z with respect to x is equal to the derivative of z with respect to y, times the derivative of y with respect to x. Expressed symbolically,

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f'(y)g'(x) \tag{7.11}$$

This rule, known as the *chain rule*, appeals easily to intuition. Given a Δx, there must result a corresponding Δy via the function $y = g(x)$, but this Δy will in turn bring about a Δz via the function $z = f(y)$. Thus there is a "chain reaction" as follows:

$$\Delta x \overset{\text{via } g}{\rightarrow} \Delta y \overset{\text{via } f}{\rightarrow} \Delta z$$

The two links in this chain entail two difference quotients, $\Delta y/\Delta x$ and $\Delta z/\Delta y$, but when they are multiplied, the Δy will cancel itself out, and we end up with

$$\frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x} = \frac{\Delta z}{\Delta x}$$

a difference quotient that relates Δz to Δx. If we take the limit of these difference quotients as $\Delta x \to 0$ (which implies $\Delta y \to 0$), each difference quotient will turn into a derivative; i.e., we shall have $(dz/dy)(dy/dx) = dz/dx$. This is precisely the result in (7.11).

In view of the function $y = g(x)$, we can express the function $z = f(y)$ as $z = f[g(x)]$, where the contiguous appearance of the two function symbols f and g indicates that this is a *composite function* (function of a function). It is for this reason that the chain rule is also referred to as the *composite-function rule* or *function-of-a-function rule*.

The extension of the chain rule to three or more functions is straightforward. If we have $z = f(y)$, $y = g(x)$, and $x = h(w)$, then

$$\frac{dz}{dw} = \frac{dz}{dy}\frac{dy}{dx}\frac{dx}{dw} = f'(y)g'(x)h'(w)$$

and similarly for cases in which more functions are involved.

Example 1 If $z = 3y^2$, where $y = 2x + 5$, then

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 6y(2) = 12y = 12(2x + 5)$$

Example 2 If $z = y - 3$, where $y = x^3$, then

$$\frac{dz}{dx} = 1(3x^2) = 3x^2$$

Example 3 The usefulness of this rule can best be appreciated when we must differentiate a function such as $z = (x^2 + 3x - 2)^{17}$. Without the chain rule at our disposal, dz/dx can be found only via the laborious route of first multiplying out the 17th-power expression. With the chain rule, however, we can take a shortcut by defining a new, *intermediate* variable $y = x^2 + 3x - 2$, so that we get in effect two functions linked in a chain:

$$z = y^{17} \qquad \text{and} \qquad y = x^2 + 3x - 2$$

The derivative dz/dx can then be found as follows:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = 17y^{16}(2x + 3) = 17(x^2 + 3x - 2)^{16}(2x + 3)$$

Example 4 Given a total-revenue function of a firm $R = f(Q)$, where output Q is a function of labor input L, or $Q = g(L)$, find dR/dL. By the chain rule, we have

$$\frac{dR}{dL} = \frac{dR}{dQ}\frac{dQ}{dL} = f'(Q)g'(L)$$

Translated into economic terms, dR/dQ is the MR function and dQ/dL is the marginal-physical-product-of-labor (MPP_L) function. Similarly, dR/dL has the connotation of the marginal-revenue-product-of-labor (MRP_L) function. Thus the result shown above constitutes the mathematical statement of the well-known result in economics that $\text{MRP}_L = \text{MR} \cdot \text{MPP}_L$.

Inverse-Function Rule

If the function $y = f(x)$ represents a one-to-one mapping, i.e., if the function is such that a different value of x will always yield a different value of y, the function f will have an *inverse function* $x = f^{-1}(y)$ (read: "x is an inverse function of y"). Here, the symbol f^{-1} is a function symbol which, like the derivative-function symbol f', signifies a function related to the function f; it does *not* mean the reciprocal of the function $f(x)$.

What the existence of an inverse function essentially means is that, in this case, not only will a given value of x yield a unique value of y [that is, $y = f(x)$], but also a given value of y will yield a unique value of x. To take a nonnumerical instance, we may exemplify the one-to-one mapping by the mapping from the set of all husbands to the set of all wives in a monogamous society. Each husband has a unique wife, and each wife has a unique husband. In contrast, the mapping from the set of all fathers to the set of all sons is not one-to-one, because a father may have more than one son, albeit each son has a unique father.

When x and y refer specifically to numbers, the property of one-to-one mapping is seen to be unique to the class of functions known as *monotonic functions*. Given a function $f(x)$, if successively larger values of the independent variable x *always* lead to successively larger values of $f(x)$, that is, if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2)$$

then the function f is said to be an *increasing* (or *monotonically increasing*) function.* If successive increases in x *always* lead to successive decreases in $f(x)$, that is, if

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) < f(x_2)$$

on the other hand, the function is said to be a *decreasing* (or *monotonically decreasing*) function. In either of these cases, an inverse function f^{-1} exists.

A practical way of ascertaining the monotonicity of a given function $y = f(x)$ is to check whether the derivative $f'(x)$ always adheres to the same algebraic sign (not zero) for all values of x. Geometrically, this means that its slope is either always upward or always downward. Thus a firm's demand curve $Q = f(P)$ that has a negative slope throughout is monotonic. As such, it has an inverse function $P = f^{-1}(Q)$, which, as mentioned previously, gives the average-revenue curve of the firm, since $P \equiv \text{AR}$.

Example 5 The function

$$y = 5x + 25$$

has the derivative $dy/dx = 5$, which is positive regardless of the value of x; thus the function is monotonic. (In this case it is increasing, because the derivative is positive.) It follows that an inverse function exists. In the present case, the inverse function is easily found by solving the given equation $y = 5x + 25$ for x. The result is the function

$$x = \tfrac{1}{5}y - 5$$

It is interesting to note that this inverse function is also monotonic, and increasing, because $dx/dy = \frac{1}{5} > 0$ for all values of y.

Generally speaking, if an inverse function exists, the original and the inverse functions must both be monotonic. Moreover, if f^{-1} is the inverse function of f, then f must be the inverse function of f^{-1}; that is, f and f^{-1} must be inverse functions of each other.

It is easy to verify that the graph of $y = f(x)$ and that of $x = f^{-1}(y)$ are one and the same, only with the axes reversed. If one lays the x axis of the f^{-1} graph over the x axis of the f graph (and similarly for the y axis), the two curves will coincide. On the other hand, if the x axis of the f^{-1} graph is laid over the y axis of

* Some writers prefer to define an *increasing function* as a function with the property that

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2) \qquad \text{[with a weak inequality]}$$

and then reserve the term *strictly increasing function* for the case where

$$x_1 > x_2 \quad \Rightarrow \quad f(x_1) > f(x_2) \qquad \text{[with a strict inequality]}$$

Under this usage, an ascending step function qualifies as an increasing (though not strictly increasing) function, despite the fact that its graph contains horizontal segments. We shall not follow this usage in the present book. Instead, we shall consider an ascending step function to be, not an increasing function, but a *nondecreasing* one. By the same token, we shall regard a descending step function not as a decreasing function, but as a *nonincreasing* one.

the f graph (and vice versa), the two curves will become *mirror images* of each other with reference to the 45° line drawn through the origin. This mirror-image relationship provides us with an easy way of graphing the inverse function f^{-1}, once the graph of the original function f is given. (You should try this with the two functions in Example 5.)

For inverse functions, the rule of differentiation is

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

This means that the derivative of the inverse function is the reciprocal of the derivative of the original function; as such, dx/dy must take the same sign as dy/dx, so that if f is increasing (decreasing), then so must be f^{-1}.

As a verification of this rule, we can refer back to Example 5, where dy/dx was found to be 5, and dx/dy equal to $\frac{1}{5}$. These two derivatives are indeed reciprocal to each other and have the same sign.

In that simple example, the inverse function is relatively easy to obtain, so that its derivative dx/dy can be found directly from the inverse function. As the next example shows, however, the inverse function is sometimes difficult to express explicitly, and thus direct differentiation may not be practicable. The usefulness of the inverse-function rule then becomes more fully apparent.

Example 6 Given $y = x^5 + x$, find dx/dy. First of all, since

$$\frac{dy}{dx} = 5x^4 + 1 > 0$$

for any value of x, the given function is monotonically increasing, and an inverse function exists. To solve the given equation for x may not be such an easy task, but the derivative of the inverse function can nevertheless be found quickly by use of the inverse-function rule:

$$\frac{dx}{dy} = \frac{1}{dy/dx} = \frac{1}{5x^4 + 1}$$

The inverse-function rule is, strictly speaking, applicable only when the function involved is a one-to-one mapping. In fact, however, we do have some leeway. For instance, when dealing with a U-shaped curve (not monotonic), we may consider the downward- and the upward-sloping segments of the curve as representing two *separate* functions, each with a restricted domain, and each being monotonic in the restricted domain. To each of these, the inverse-function rule can then again be applied.

EXERCISE 7.3

1 Given $y = u^3 + 1$, where $u = 5 - x^2$, find dy/dx by the chain rule.

2 Given $w = ay^2$ and $y = bx^2 + cx$, find dw/dx by the chain rule.

3 Use the chain rule to find dy/dx for the following:
(a) $y = (3x^2 - 13)^3$ (b) $y = (8x^3 - 5)^9$ (c) $y = (ax + b)^4$

4 Given $y = (16x + 3)^{-2}$, use the chain rule to find dy/dx. Then rewrite the function as $y = 1/(16x + 3)^2$ and find dy/dx by the quotient rule. Are the answers identical?

5 Given $y = 7x + 21$, find its inverse function. Then find dy/dx and dx/dy, and verify the inverse-function rule. Also verify that the graphs of the two functions bear a mirror-image relationship to each other.

6 Are the following functions monotonic?
(a) $y = -x^6 + 5 \quad (x > 0)$ (b) $y = 4x^5 + x^3 + 3x$
For each monotonic function, find dx/dy by the inverse-function rule.

7.4 PARTIAL DIFFERENTIATION

Hitherto, we have considered only the derivatives of functions of a single independent variable. In comparative-static analysis, however, we are likely to encounter the situation in which several parameters appear in a model, so that the equilibrium value of each endogenous variable may be a function of more than one parameter. Therefore, as a final preparation for the application of the concept of derivative to comparative statics, we must learn how to find the derivative of a function of more than one variable.

Partial Derivatives

Let us consider a function

$$y = f(x_1, x_2, \ldots, x_n) \tag{7.12}$$

where the variables x_i $(i = 1, 2, \ldots, n)$ are all *independent* of one another, so that each can vary by itself without affecting the others. If the variable x_1 undergoes a change Δx_1 while $x_2, \ldots, x_n$ all remain fixed, there will be a corresponding change in y, namely, Δy. The difference quotient in this case can be expressed as

$$\frac{\Delta y}{\Delta x_1} = \frac{f(x_1 + \Delta x_1, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{\Delta x_1} \tag{7.13}$$

If we take the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \to 0$, that limit will constitute a derivative. We call it the *partial derivative* of y with respect to x_1, to indicate that all the other independent variables in the function are held constant when taking this particular derivative. Similar partial derivatives can be defined for infinitesimal changes in the other independent variables. The process of taking partial derivatives is called *partial differentiation*.

Partial derivatives are assigned distinctive symbols. In lieu of the letter d (as in dy/dx), we employ the symbol ∂, which is a variant of the Greek δ (lower case delta). Thus we shall now write $\partial y/\partial x_i$, which is read: "the partial derivative of y

with respect to x_i." The partial-derivative symbol sometimes is also written as $\frac{\partial}{\partial x_i} y$; in that case, its $\partial/\partial x_i$ part can be regarded as an operator symbol instructing us to take the partial derivative of (some function) with respect to the variable x_i. Since the function involved here is denoted in (7.12) by f, it is also permissible to write $\partial f/\partial x_i$.

Is there also a partial-derivative counterpart for the symbol $f'(x)$ that we used before? The answer is yes. Instead of f', however, we now use f_1, f_2, etc., where the subscript indicates which independent variable (alone) is being allowed to vary. If the function in (7.12) happens to be written in terms of unsubscripted variables, such as $y = f(u, v, w)$, then the partial derivatives may be denoted by f_u, f_v, and f_w rather than f_1, f_2, and f_3.

In line with these notations, and on the basis of (7.12) and (7.13), we can now define

$$f_1 \equiv \frac{\partial y}{\partial x_1} \equiv \lim_{\Delta x_1 \to 0} \frac{\Delta y}{\Delta x_1}$$

as the first in the set of n partial derivatives of the function f.

Techniques of Partial Differentiation

Partial differentiation differs from the previously discussed differentiation primarily in that we must hold $(n - 1)$ independent variables *constant* while allowing *one* variable to vary. Inasmuch as we have learned how to handle *constants* in differentiation, the actual differentiation should pose little problem.

Example 1 Given $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$, find the partial derivatives. When finding $\partial y/\partial x_1$ (or f_1), we must bear in mind that x_2 is to be treated as a constant during differentiation. As such, x_2 will drop out in the process if it is an *additive* constant (such as the term $4x_2^2$) but will be retained if it is a *multiplicative* constant (such as in the term x_1x_2). Thus we have

$$\frac{\partial y}{\partial x_1} \equiv f_1 = 6x_1 + x_2$$

Similarly, by treating x_1 as a constant, we find that

$$\frac{\partial y}{\partial x_2} \equiv f_2 = x_1 + 8x_2$$

Note that, like the primitive function f, both partial derivatives are themselves functions of the variables x_1 and x_2. That is, we may write them as two derived functions

$$f_1 = f_1(x_1, x_2) \qquad \text{and} \qquad f_2 = f_2(x_1, x_2)$$

For the point $(x_1, x_2) = (1, 3)$ in the domain of the function f, for example, the partial derivatives will take the following specific values:

$$f_1(1, 3) = 6(1) + 3 = 9 \qquad \text{and} \qquad f_2(1, 3) = 1 + 8(3) = 25$$

Example 2 Given $y = f(u, v) = (u + 4)(3u + 2v)$, the partial derivatives can be found by use of the product rule. By holding v constant, we have

$$f_u = (u + 4)(3) + 1(3u + 2v) = 2(3u + v + 6)$$

Similarly, by holding u constant, we find that

$$f_v = (u + 4)(2) + 0(3u + 2v) = 2(u + 4)$$

When $u = 2$ and $v = 1$, these derivatives will take the following values:

$$f_u(2, 1) = 2(13) = 26 \qquad \text{and} \qquad f_v(2, 1) = 2(6) = 12$$

Example 3 Given $y = (3u - 2v)/(u^2 + 3v)$, the partial derivatives can be found by use of the quotient rule:

$$\frac{\partial y}{\partial u} = \frac{3(u^2 + 3v) - 2u(3u - 2v)}{(u^2 + 3v)^2} = \frac{-3u^2 + 4uv + 9v}{(u^2 + 3v)^2}$$

$$\frac{\partial y}{\partial v} = \frac{-2(u^2 + 3v) - 3(3u - 2v)}{(u^2 + 3v)^2} = \frac{-u(2u + 9)}{(u^2 + 3v)^2}$$

Geometric Interpretation of Partial Derivatives

As a special type of derivative, a partial derivative is a measure of the instantaneous rates of change of some variable, and in that capacity it again has a geometric counterpart in the slope of a particular curve.

Let us consider a production function $Q = Q(K, L)$, where Q, K, and L denote output, capital input, and labor input, respectively. This function is a particular two-variable version of (7.12), with $n = 2$. We can therefore define two partial derivatives $\partial Q/\partial K$ (or Q_K) and $\partial Q/\partial L$ (or Q_L). The partial derivative Q_K relates to the rates of change in output with respect to infinitesimal changes in capital, while labor input is held constant. Thus Q_K symbolizes the marginal-physical-product-of-capital (MPP_K) function. Similarly, the partial derivative Q_L is the mathematical representation of the MPP_L function.

Geometrically, the production function $Q = Q(K, L)$ can be depicted by a *production surface* in a 3-space, such as is shown in Fig. 7.4. The variable Q is plotted vertically, so that for any point (K, L) in the base plane (KL plane), the height of the surface will indicate the output Q. The domain of the function should consist of the entire nonnegative quadrant of the base plane, but for our purposes it is sufficient to consider a subset of it, the rectangle OK_0BL_0. As a consequence, only a small portion of the production surface is shown in the figure.

Let us now hold capital fixed at the level K_0 and consider only variations in the input L. By setting $K = K_0$, all points in our (curtailed) domain become irrelevant except those on the line segment K_0B. By the same token, only the curve K_0CDA (a cross section of the production surface) will be germane to the present discussion. This curve represents a total-physical-product-of-labor (TPP_L)

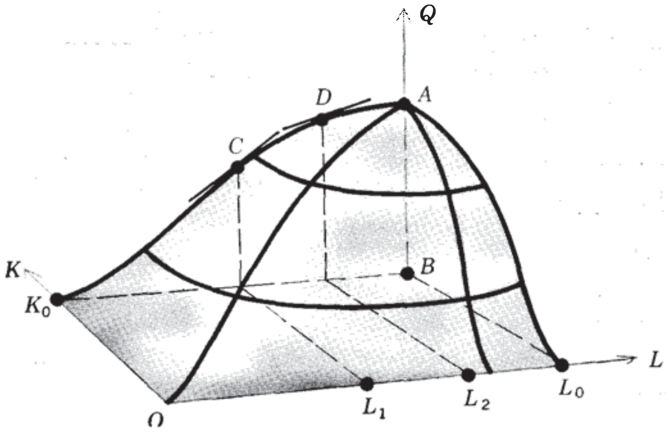

Figure 7.4

curve for a fixed amount of capital $K = K_0$; thus we may read from its slope the rate of change of Q with respect to changes in L while K is held constant. It is clear, therefore, that the slope of a curve such as K_0CDA represents the geometric counterpart of the partial derivative Q_L. Once again, we note that the slope of a total (TPP_L) curve is its corresponding marginal ($\text{MPP}_L \equiv Q_L$) curve.

It was mentioned earlier that a partial derivative is a function of all the independent variables of the primitive function. That Q_L is a function of L is immediately obvious from the K_0CDA curve itself. When $L = L_1$, the value of Q_L is equal to the slope of the curve at point C; but when $L = L_2$, the relevant slope is the one at point D. Why is Q_L also a function of K? The answer is that K can be fixed at various levels, and for each fixed level of K, there will result a different TPP_L curve (a different cross section of the production surface), with inevitable repercussions on the derivative Q_L. Hence Q_L is also a function of K.

An analogous interpretation can be given to the partial derivative Q_K. If the labor input is held constant instead of K (say, at the level of L_0), the line segment L_0B will be the relevant subset of the domain, and the curve L_0A will indicate the relevant subset of the production surface. The partial derivative Q_K can then be interpreted as the slope of the curve L_0A—bearing in mind that the K axis extends from southeast to northwest in Fig. 7.4. It should be noted that Q_K is again a function of both the variables L and K.

EXERCISE 7.4

1 Find $\partial y/\partial x_1$ and $\partial y/\partial x_2$ for each of the following functions:

(*a*) $y = 2x_1^3 - 11x_1^2x_2 + 3x_2^2$ (*c*) $y = (2x_1 + 3)(x_2 - 2)$

(*b*) $y = 7x_1 + 5x_1x_2^2 - 9x_2^3$ (*d*) $y = (4x_1 + 3)/(x_2 - 2)$

2 Find f_x and f_y from the following:

(a) $f(x, y) = x^2 + 5xy - y^3$ (c) $f(x, y) = \dfrac{2x - 3y}{x + y}$

(b) $f(x, y) = (x^2 - 3y)(x - 2)$ (d) $f(x, y) = \dfrac{x^2 - 1}{xy}$

3 From the answers to the preceding problem, find $f_x(1,2)$—the value of the partial derivative f_x when $x = 1$ and $y = 2$—for each function.

4 Given the production function $Q = 96K^{0.3}L^{0.7}$, find the MPP_K and MPP_L functions. Is MPP_K a function of K alone, or of both K and L? What about MPP_L?

5 If the utility function of an individual takes the form

$$U = U(x_1, x_2) = (x_1 + 2)^2(x_2 + 3)^3$$

where U is total utility, and x_1 and x_2 are the quantities of two commodities consumed:

(a) Find the marginal-utility function of each of the two commodities.

(b) Find the value of the marginal utility of the first commodity when 3 units of each commodity are consumed.

7.5 APPLICATIONS TO COMPARATIVE-STATIC ANALYSIS

Equipped with the knowledge of the various rules of differentiation, we can at last tackle the problem posed in comparative-static analysis: namely, how the equilibrium value of an endogenous variable will change when there is a change in any of the exogenous variables or parameters.

Market Model

First let us consider again the simple one-commodity market model of (3.1). That model can be written in the form of two equations:

$$Q = a - bP \qquad (a, b > 0) \qquad \text{[demand]}$$

$$Q = -c + dP \qquad (c, d > 0) \qquad \text{[supply]}$$

with solutions

$$\bar{P} = \frac{a + c}{b + d} \tag{7.14}$$

$$\bar{Q} = \frac{ad - bc}{b + d} \tag{7.15}$$

These solutions will be referred to as being in the *reduced form*: the two endogenous variables have been reduced to explicit expressions of the four mutually independent parameters a, b, c, and d.

To find how an infinitesimal change in one of the parameters will affect the value of $\bar{P}$, one has only to differentiate (7.14) partially with respect to each of the parameters. If the *sign* of a partial derivative, say, $\partial\bar{P}/\partial a$, can be determined from the given information about the parameters, we shall know the direction in which $\bar{P}$ will move when the parameter a changes; this constitutes a qualitative conclusion. If the magnitude of $\partial\bar{P}/\partial a$ can be ascertained, it will constitute a quantitative conclusion.

Similarly, we can draw qualitative or quantitative conclusions from the partial derivatives of $\bar{Q}$ with respect to each parameter, such as $\partial\bar{Q}/\partial a$. To avoid misunderstanding, however, a clear distinction should be made between the two derivatives $\partial\bar{Q}/\partial a$ and $\partial Q/\partial a$. The latter derivative is a concept appropriate to the demand function taken alone, and without regard to the supply function. The derivative $\partial\bar{Q}/\partial a$ pertains, on the other hand, to the equilibrium quantity in (7.15) which, being in the nature of a solution of the model, takes into account the interaction of demand and supply together. To emphasize this distinction, we shall refer to the partial derivatives of $\bar{P}$ and $\bar{Q}$ with respect to the parameters as *comparative-static derivatives*.

Concentrating on $\bar{P}$ for the time being, we can get the following four partial derivatives from (7.14):

$$\frac{\partial\bar{P}}{\partial a} = \frac{1}{b+d} \qquad \left[\text{parameter } a \text{ has the coefficient } \frac{1}{b+d}\right]$$

$$\frac{\partial\bar{P}}{\partial b} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2} \qquad [\text{quotient rule}]$$

$$\frac{\partial\bar{P}}{\partial c} = \frac{1}{b+d}\left(= \frac{\partial\bar{P}}{\partial a}\right)$$

$$\frac{\partial\bar{P}}{\partial d} = \frac{0(b+d) - 1(a+c)}{(b+d)^2} = \frac{-(a+c)}{(b+d)^2}\left(= \frac{\partial\bar{P}}{\partial b}\right)$$

Since all the parameters are restricted to being positive in the present model, we can conclude that

$$(7.16) \qquad \frac{\partial\bar{P}}{\partial a} = \frac{\partial\bar{P}}{\partial c} > 0 \qquad \text{and} \qquad \frac{\partial\bar{P}}{\partial b} = \frac{\partial\bar{P}}{\partial d} < 0$$

For a fuller appreciation of the results in (7.16), let us look at Fig. 7.5, where each diagram shows a change in *one* of the parameters. As before, we are plotting Q (rather than P) on the vertical axis.

Figure 7.5*a* pictures an increase in the parameter a (to a'). This means a higher vertical intercept for the demand curve, and inasmuch as the parameter b (the slope parameter) is unchanged, the increase in a results in a parallel upward shift of the demand curve from D to D'. The intersection of D' and the supply curve S determines an equilibrium price $\bar{P}'$, which is greater than the old equilibrium price $\bar{P}$. This corroborates the result that $\partial\bar{P}/\partial a > 0$, although for the sake of exposition we have shown in Fig. 7.5*a* a much larger change in the parameter a than what the concept of derivative implies.

The situation in Fig. 7.5*c* has a similar interpretation; but since the increase takes place in the parameter c, the result is a parallel shift of the supply curve instead. Note that this shift is downward because the supply curve has a vertical intercept of $-c$; thus an increase in c would mean a change in the intercept, say, from -2 to -4. The graphical comparative-static result, that $\overline{P}'$ exceeds $\overline{P}$, again conforms to what the positive sign of the derivative $\partial\overline{P}/\partial c$ would lead us to expect.

Figures 7.5*b* and 7.5*d* illustrate the effects of changes in the slope parameters b and d of the two functions in the model. An increase in b means that the slope of the demand curve will assume a larger numerical (absolute) value; i.e., it will become steeper. In accordance with the result $\partial\overline{P}/\partial b < 0$, we find a decrease in $\overline{P}$ in this diagram. The increase in d that makes the supply curve steeper also results in a decrease in the equilibrium price. This is, of course, again in line with the negative sign of the comparative-static derivative $\partial\overline{P}/\partial d$.

Thus far, all the results in (7.16) seem to have been obtainable graphically. If so, why should we bother to learn differentiation at all? The answer is that the differentiation approach has at least two major advantages. First, the graphical technique is subject to a dimensional restriction, but differentiation is not. Even

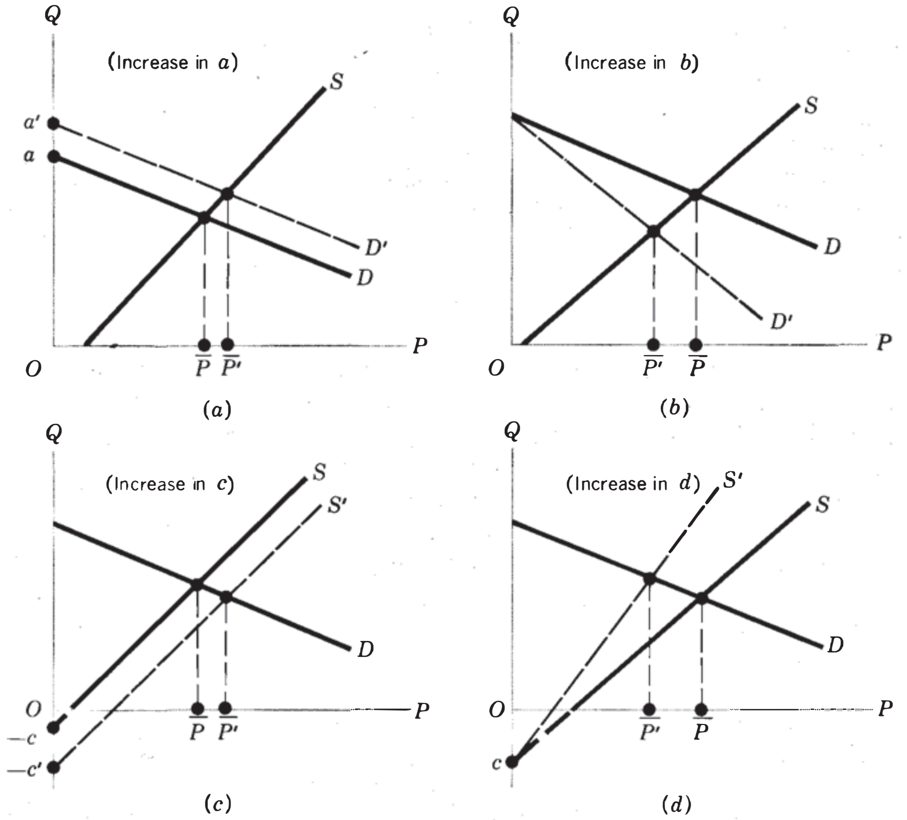

Figure 7.5

when the number of endogenous variables and parameters is such that the equilibrium state cannot be shown graphically, we can nevertheless apply the differentiation techniques to the problem. Second, the differentiation method can yield results that are on a higher level of generality. The results in (7.16) will remain valid, regardless of the specific values that the parameters a, b, c, and d take, as long as they satisfy the sign restrictions. So the comparative-static conclusions of this model are, in effect, applicable to an infinite number of combinations of (linear) demand and supply functions. In contrast, the graphical approach deals only with some specific members of the family of demand and supply curves, and the analytical result derived therefrom is applicable, strictly speaking, only to the specific functions depicted.

The above serves to illustrate the application of partial differentiation to comparative-static analysis of the simple market model, but only half of the task has actually been accomplished, for we can also find the comparative-static derivatives pertaining to $\bar{Q}$. This we shall leave to you as an exercise.

National-Income Model

In place of the simple national-income model discussed in Chap. 3, let us study a slightly enlarged model with three endogenous variables, Y (national income), C (consumption), and T (taxes):

$$\begin{aligned} & Y = C + I_0 + G_0 & \\ (7.17)\qquad & C = \alpha + \beta(Y - T) & (\alpha > 0; \quad 0 < \beta < 1) \\ & T = \gamma + \delta Y & (\gamma > 0; \quad 0 < \delta < 1) \end{aligned}$$

The first equation in this system gives the equilibrium condition for national income, while the second and third equations show, respectively, how C and T are determined in the model.

The restrictions on the values of the parameters α, β, γ, and δ can be explained thus: α is positive because consumption is positive even if disposable income $(Y - T)$ is zero; β is a positive fraction because it represents the marginal propensity to consume; γ is positive because even if Y is zero the government will still have a positive tax revenue (from tax bases other than income); and finally, δ is a positive fraction because it represents an income tax rate, and as such it cannot exceed 100 percent. The exogenous variables I_0 (investment) and G_0 (government expenditure) are, of course, nonnegative. All the parameters and exogenous variables are assumed to be independent of one another, so that any one of them can be assigned a new value without affecting the others.

This model can be solved for $\bar{Y}$ by substituting the third equation of (7.17) into the second and then substituting the resulting equation into the first. The equilibrium income (in reduced form) is

$$(7.18)\qquad \bar{Y} = \frac{\alpha - \beta\gamma + I_0 + G_0}{1 - \beta + \beta\delta}$$

Similar equilibrium values can also be found for the endogenous variables C and T, but we shall concentrate on the equilibrium income.

From (7.18), there can be obtained six comparative-static derivatives. Among these, the following three have special policy significance:

$$\frac{\partial \overline{Y}}{\partial G_0} = \frac{1}{1 - \beta + \beta\delta} > 0 \tag{7.19}$$

$$\frac{\partial \overline{Y}}{\partial \gamma} = \frac{-\beta}{1 - \beta + \beta\delta} < 0 \tag{7.20}$$

$$\frac{\partial \overline{Y}}{\partial \delta} = \frac{-\beta(\alpha - \beta\gamma + I_0 + G_0)}{(1 - \beta + \beta\delta)^2} = \frac{-\beta\overline{Y}}{1 - \beta + \beta\delta} < 0 \qquad [\text{by } (7.18)] \tag{7.21}$$

The partial derivative in (7.19) gives us the *government-expenditure multiplier*. It has a positive sign here because β is less than 1, and $\beta\delta$ is greater than zero. If numerical values are given for the parameters β and δ, we can also find the numerical value of this multiplier from (7.19). The derivative in (7.20) may be called the *nonincome-tax multiplier*, because it shows how a change in γ, the government revenue from nonincome-tax sources, will affect the equilibrium income. This multiplier is negative in the present model because the denominator in (7.20) is positive and the numerator is negative. Lastly, the partial derivative in (7.21) represents an *income-tax-rate multiplier*. For any positive equilibrium income, this multiplier is also negative in the model.

Again, note the difference between the two derivatives $\partial\overline{Y}/\partial G_0$ and $\partial Y/\partial G_0$. The former is derived from (7.18), the expression for the equilibrium income. The latter, obtainable from the first equation in (7.17), is $\partial Y/\partial G_0 = 1$, which is altogether different in magnitude and in concept.

Input-Output Model

The solution of an open input-output model appears as a matrix equation $\overline{x} = (I - A)^{-1}d$. If we denote the inverse matrix $(I - A)^{-1}$ by $B = [b_{ij}]$, then, for instance, the solution for a three-industry economy can be written as $\overline{x} = Bd$, or

$$\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{7.22}$$

What will be the rates of change of the solution values $\overline{x}_j$ with respect to the exogenous final demands d_1, d_2, and d_3? The general answer is that

$$\frac{\partial \overline{x}_j}{\partial d_k} = b_{jk} \qquad (j, k = 1, 2, 3) \tag{7.23}$$

To see this, let us multiply out Bd in (7.22) and express the solution as

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11}d_1 + b_{12}d_2 + b_{13}d_3 \\ b_{21}d_1 + b_{22}d_2 + b_{23}d_3 \\ b_{31}d_1 + b_{32}d_2 + b_{33}d_3 \end{bmatrix}$$

In this system of three equations, each one gives a particular solution value as a function of the exogenous final demands. Partial differentiation of these will produce a total of nine comparative-static derivatives:

$$(7.23') \qquad \begin{matrix} \dfrac{\partial \bar{x}_1}{\partial d_1} = b_{11} & \dfrac{\partial \bar{x}_1}{\partial d_2} = b_{12} & \dfrac{\partial \bar{x}_1}{\partial d_3} = b_{13} \\[2ex] \dfrac{\partial \bar{x}_2}{\partial d_1} = b_{21} & \dfrac{\partial \bar{x}_2}{\partial d_2} = b_{22} & \dfrac{\partial \bar{x}_2}{\partial d_3} = b_{23} \\[2ex] \dfrac{\partial \bar{x}_3}{\partial d_1} = b_{31} & \dfrac{\partial \bar{x}_3}{\partial d_2} = b_{32} & \dfrac{\partial \bar{x}_3}{\partial d_3} = b_{33} \end{matrix}$$

This is simply the expanded version of (7.23).

Reading (7.23′) as three distinct columns, we may combine the three derivatives in each column into a matrix (vector) derivative:

$$(7.23'') \qquad \frac{\partial \bar{x}}{\partial d_1} \equiv \frac{\partial}{\partial d_1}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \qquad \frac{\partial \bar{x}}{\partial d_2} = \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} \qquad \frac{\partial \bar{x}}{\partial d_3} = \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$$

Since the three column vectors in (7.23″) are merely the columns of the matrix B, by further consolidation we can summarize the nine derivatives in a single matrix derivative $\partial \bar{x}/\partial d$. Given $\bar{x} = Bd$, we can simply write

$$\frac{\partial \bar{x}}{\partial d} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = B$$

This is a compact way of denoting all the comparative-static derivatives of our open input-output model. Obviously, this matrix derivative can easily be extended from the present three-industry model to the general n-industry case.

Comparative-static derivatives of the input-output model are useful as tools of economic planning, for they provide the answer to the question: If the planning targets, as reflected in $(d_1, d_2, \ldots, d_n)$, are revised, and if we wish to take care of all direct and indirect requirements in the economy so as to be completely free of bottlenecks, how must we change the output goals of the n industries?

EXERCISE 7.5

1 Examine the comparative-static properties of the equilibrium quantity in (7.15), and check your results by graphic analysis.

2 On the basis of (7.18), find the partial derivatives $\partial\bar{Y}/\partial I_0$, $\partial\bar{Y}/\partial\alpha$, and $\partial\bar{Y}/\partial\beta$. Interpret their meanings and determine their signs.

3 The numerical input-output model (5.21) was solved in Sec. 5.7.
 (*a*) How many comparative-static derivatives can be derived?
 (*b*) Write out these derivatives in the form of (7.23′) and (7.23″).

7.6 NOTE ON JACOBIAN DETERMINANTS

The study of partial derivatives above was motivated solely by comparative-static considerations. But partial derivatives also provide a means of testing whether there exists functional (linear *or* nonlinear) dependence among a set of n functions in n variables. This is related to the notion of Jacobian determinants (named after Jacobi).

Consider the two functions

$$\begin{aligned} y_1 &= 2x_1 + 3x_2 \\ y_2 &= 4x_1^2 + 12x_1x_2 + 9x_2^2 \end{aligned} \tag{7.24}$$

If we get all the four partial derivatives

$$\frac{\partial y_1}{\partial x_1} = 2 \qquad \frac{\partial y_1}{\partial x_2} = 3 \qquad \frac{\partial y_2}{\partial x_1} = 8x_1 + 12x_2 \qquad \frac{\partial y_2}{\partial x_2} = 12x_1 + 18x_2$$

and arrange them into a square matrix in a prescribed order, called a Jacobian matrix and denoted by J, and then take its determinant, the result will be what is known as a *Jacobian determinant* (or a *Jacobian*, for short), denoted by $|J|$:

$$|J| \equiv \begin{vmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ (8x_1 + 12x_2) & (12x_1 + 18x_2) \end{vmatrix} \tag{7.25}$$

For economy of space, this Jacobian is sometimes also expressed as

$$|J| \equiv \left| \frac{\partial(y_1, y_2)}{\partial(x_1, x_2)} \right|$$

More generally, if we have n differentiable functions in n variables, not necessarily linear,

$$\begin{aligned} y_1 &= f^1(x_1, x_2, \ldots, x_n) \\ y_2 &= f^2(x_1, x_2, \ldots, x_n) \\ &\cdots\cdots\cdots\cdots\cdots \\ y_n &= f^n(x_1, x_2, \ldots, x_n) \end{aligned} \tag{7.26}$$

where the symbol f^n denotes the nth function (and *not* the function raised to the nth power), we can derive a total of n^2 partial derivatives. Together, they will give rise to the Jacobian

$$|J| \equiv \left| \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \right| \equiv \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix} \tag{7.27}$$

A Jacobian test for the existence of functional dependence among a set of n functions is provided by the following theorem: The Jacobian $|J|$ defined in (7.27) will be identically zero for all values of $x_1, \ldots, x_n$ if and only if the n functions $f^1, \ldots, f^n$ in (7.26) are functionally (linearly or nonlinearly) dependent.

As an example, for the two functions in (7.24) the Jacobian as given in (7.25) has the value

$$|J| = (24x_1 + 36x_2) - (24x_1 + 36x_2) = 0$$

That is, the Jacobian vanishes for all values of x_1 and x_2. Therefore, according to the theorem, the two functions in (7.24) must be dependent. You can verify that y_2 is simply y_1 squared; thus they are indeed functionally dependent—here *non*linearly dependent.

Let us now consider the special case of *linear* functions. We have earlier shown that the rows of the coefficient matrix A of a linear-equation system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= d_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= d_2 \\ &\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= d_n \end{aligned} \tag{7.28}$$

are linearly dependent if and only if the determinant $|A| = 0$. This result can now be interpreted as a special application of the Jacobian criterion of functional dependence.

Take the left side of each equation in (7.28) as a separate function of the n variables $x_1, \ldots, x_n$, and denote these functions by $y_1, \ldots, y_n$. The partial derivatives of these functions will turn out to be $\partial y_1/\partial x_1 = a_{11}$, $\partial y_1/\partial x_2 = a_{12}$, etc., so that we may write, in general, $\partial y_i/\partial x_j = a_{ij}$. In view of this, the elements of the Jacobian of these n functions will be precisely the elements of the coefficient matrix A, already arranged in the correct order. That is, we have $|J| = |A|$, and thus the Jacobian criterion of functional dependence among $y_1, \ldots, y_n$—or, what amounts to the same thing, functional dependence among the rows of the coefficient matrix A—is equivalent to the criterion $|A| = 0$ in the present linear case.

In the above, the Jacobian was discussed in the context of a system of n functions in n variables. It should be pointed out, however, that the Jacobian in (7.27) is defined even if each function in (7.26) contains more than n variables, say, $n + 2$ variables:

$$y_i = f^i(x_1, \ldots, x_n, x_{n+1}, x_{n+2}) \qquad (i = 1, 2, \ldots, n)$$

In such a case, if we hold any two of the variables (say, x_{n+1} and x_{n+2}) constant, or treat them as parameters, we will again have n functions in exactly n variables and can form a Jacobian. Moreover, by holding a different pair of the x variables constant, we can form a different Jacobian. Such a situation will indeed be encountered in Chap. 8 in connection with the discussion of the implicit-function theorem.

EXERCISE 7.6

1 Use Jacobian determinants to test the existence of functional dependence between the functions paired below:

(*a*) $y_1 = 3x_1^2 + x_2$
$y_2 = 9x_1^4 + 6x_1^2(x_2 + 4) + x_2(x_2 + 8) + 12$

(*b*) $y_1 = 3x_1^2 + 2x_2^2$
$y_2 = 5x_1 + 1$

2 Consider (7.22) as a set of three functions $\bar{x}_i = f^i(d_1, d_2, d_3)$ (with $i = 1, 2, 3$).

(*a*) Write out the 3×3 Jacobian. Does it have some relation to (7.23′)? Can we write $|J| = |B|$?

(*b*) Since $B \equiv (I - A)^{-1}$, can we conclude that $|B| \neq 0$? What can we infer from this about the three equations in (7.22)?